ANALOG COMPUTER FUNDAMENTALS

With an Introduction to Matrix Programming Methods

by

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Ann Arbor, Michigan

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Chapter 1. BASIC ANALOG BLOCKS

1.1 Common Uses of the Analog Computer

Although analog computers may be used for the solution of problems which do not involve differential equations, they are often considered as differential equation solvers. For this reason, the analog computer has been called the "differential analyzer."

As we know, differential equations are important in the study of dynamic systems because in these systems the important variables are changing with respect to each other, so that their behavior may be expressed mathematically by differential equations. In some cases the physical variables are changing with respect to real time, and we are asked to find the so called "transient behavior" of the system. In other cases the variables are not changing with respect to time, and we may want to study the "steady-state" solution of the system. Although the analog computer may be used for steady-state problems as well as for transient problems, it is in the latter case in which the computer has been used more frequently.

The analog computer has been found useful in the solution of ordinary differential equations with constant coefficients such as the equation

$$a_1 \frac{dy}{dt} + a_2 \frac{d^2y}{dt^2} + \ldots + a_n \frac{d^ny}{dt^n} = F \tag{1.1.1}$$

where the coefficients $a_1,\ldots,a_n$ are constant, and where F is a forcing function which may be either zero or some arbitrary function of t. In fact, it is not much harder to generalize the forcing function F to the form

$$F = f(t, \frac{dy}{dt},\ldots, \frac{d^my}{dt^m}) \qquad m \leq n. \tag{1.1.2}$$

That is, the right hand side of the equation may be a function of the independent variable t and one or more of the derivatives.

The computer may also be used for solving linear differential equations with variable coefficients such as

$$f_1(t) \frac{dy}{dt} + \ldots + f_n(t) \frac{d^ny}{dt^n} = F(t), \tag{1.1.3}$$

where the functions $f_i(t)$ are functions of time rather than constants.

One of the most useful applications of the analog computer is in the solution of non-linear equations, which in general do not lend themselves to analytic solution. As we recall, a non-linear differential equation is one in which the dependent variable or one or more of its derivatives appear raised to a power other than unity or as the argument of a function. Examples of non-linear ordinary differential equations are:

$$\frac{d^2x}{dt^2} - \epsilon(1-x^2)\frac{dx}{dt} + x = 0$$

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + K_1 \sin y = K_2$$

$$\frac{d^2x}{dt^2} + x^c = 0.$$

Certain types of partial differential equations, linear and non-linear, can also be solved with the electronic analog computer. The most important are:

a) Parabolic equations, such as the "diffusion equation"

$$\nabla^2\phi = k_1 \frac{\partial\phi}{\partial t} + k_2\phi + k_3,$$

which is important in the study of heat transfer.

b) Hyperbolic equations, such as the wave equation

$$\nabla^2\phi = k_1 \frac{\partial^2\phi}{\partial t^2} + k_2 \frac{\partial\phi}{\partial t} + k_3\phi ,$$

which governs many physical systems involving the propagation of waves.

Elliptic equations such as Poisson's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = k$$

can be solved more conveniently with less expensive passive analog networks and are seldom treated with electronic analog computers.

1.2 The Components of an Analog Computer

Suppose that the equation

$$a_0 y + f_1(t)\cdot\frac{dy}{dt} - a_2\frac{d^2y}{dt^2} = 0$$

is to be solved with an electronic analog computer. By observing the operations required in this equation, we conclude that components are needed to:

a) multiply a variable or a function times a constant, as in

$$a_0 y , \quad a_2\frac{d^2y}{dt^2} , \text{ etc.},$$

b) generate the derivatives of a variable,

c) generate arbitrary functions such as $f_1(t)$,

d) multiply two functions such as $f_1(t)$ and $\frac{dy}{dt}$,

e) perform the addition and subtraction of variables and functions as required by the left-hand side of the equation.

In an electronic analog computer these basic operations are performed on voltages by electronic "black boxes" which accept voltages at their input terminals, operate on these voltages continuously, and produce voltages at their output terminals that are some function of the input voltages.

The analogy involved in the use of this computer is the analogy which the user must set up between the physical variables such as displacements, velocities, angles, pressures and the various voltages present in the computer. The procedure of setting up this analogy is very simple, and will be explained in Chapter 2. Before this is done, however, we must study the individual computer components which perform the required fundamental operations.

1.3 Multiplication by a Constant

One of the simplest operations which may be performed on a voltage is multiplying it by a constant. If the constant is less than 1, the operation may be performed with a potentiometer or attenuator (sometimes called a "pot") as shown in Fig. 1.3.1

The circuit diagram in Fig. 1.3.1(a) shows that if a voltage $E_1(t)$ is applied at the input terminal, then any fraction K of this voltage may be transmitted to the output terminal by moving the slider up or down. The fraction K is always less than 1, and may be indicated by a calibrated dial which is connected to the slider. A diagram of the dial is shown in Fig. 1.3.1(b).

It is more convenient to think of the potentiometer in terms of its block diagram shown in Fig. 1.3.1(c). This is a one-line diagram (we do not show the third terminal which is connected to the machine reference ground) which conveys the idea of the operation performed without any reference to circuit details.

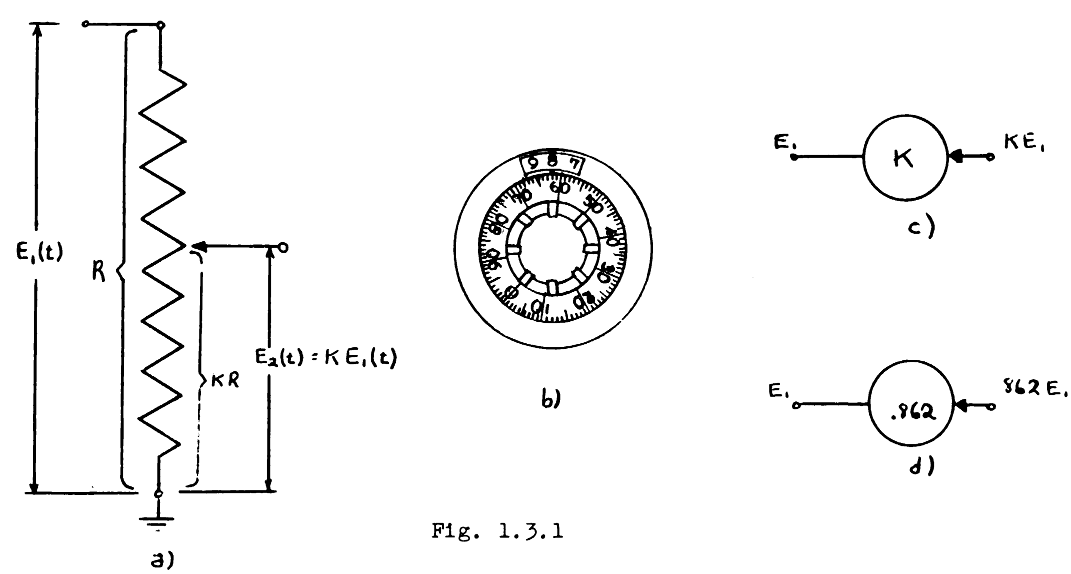


Fig. 1.3.1

The potentiometers used in most analog computers have a resistance element which is a helix of ten turns. The turn counter in the dial displays the turn number in a small window. This turn number corresponds to the tenth digits of K. The face of the dial is divided into ten parts which represent the hundredth digits of K, and each of these is divided into ten parts representing the thousandth digits of K. In this manner, the constant K may be set with an accuracy of one part in a thousand. Fig. 1.3.1d shows the block diagram of the "pot" corresponding to the dial setting shown in Fig. 1.3.1b.

The input and output voltages $E_1(t)$ and $E_2(t)$ are in general functions of time. In the condensed diagrams of Figs. 1.3.1c and d the symbol indicating time dependence (t) is dropped for simplicity, but it is always implied. The same thing will be done in some of the block diagrams in these notes.

The potentiometer is a passive device and can only provide attenuation, that is, the constant K can never be greater than 1. An active device which provides both attenuation and amplification is the constant multiplier block of Fig. 1.3.2. Figure 1.3.2a shows how this block is implemented from an amplifier, usually

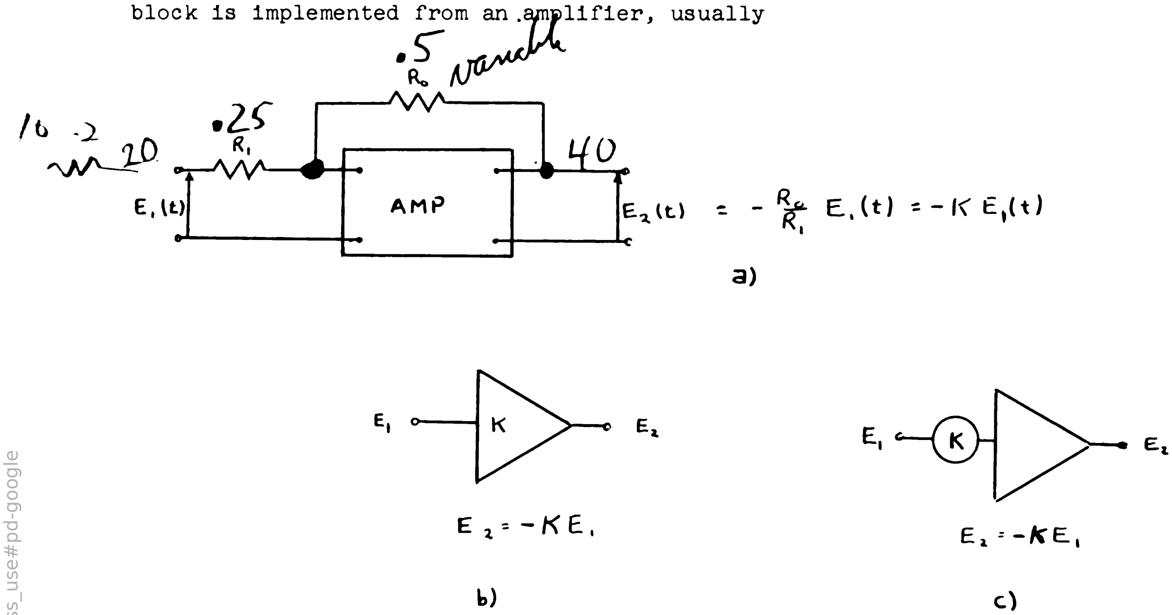


Fig. 1.3.2

called an "operational amplifier," and two resistances R_0 and R_1. The operational equation to the right of the diagram indicates that the output voltage is equal to the negative of the input voltage times a constant equal to the ratio R_0/R_1. This ratio may be called the gain of the block, and it may be adjusted to values greater than as well as less than 1. If this gain is K, the one-line diagrams of Fig. 1.3.2b and c are more convenient to use and are less circuit oriented. The notation of Fig. 1.3.2c will be used in these notes.

It should be noticed that the circle and the triangle of Fig. 1.3.2c form a symbol and should not be confused with the symbol of an operational amplifier in series with a potentiometer. The figure below shows the difference between these symbols.

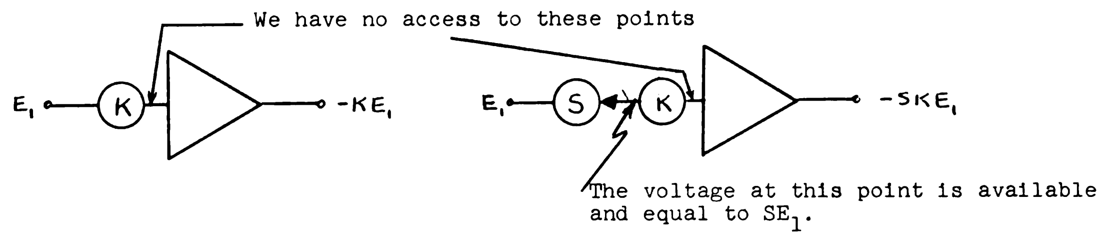


Fig. 1.3.3 shows four examples of the use of the "constant multiplier." In Fig. 1.3.3a a constant voltage of 5 machine units (5 volts) is multiplied by a gain of 3, and reversed in sign, to produce a constant output of -15 machine units. In Fig. 1.3.3b the input voltage is negative and the gain is less than unity. In Fig. 1.3.3c the time-varying function F(t) is multiplied by -a, and in Fig. 1.3.3d the analog block is used to change the sign of the

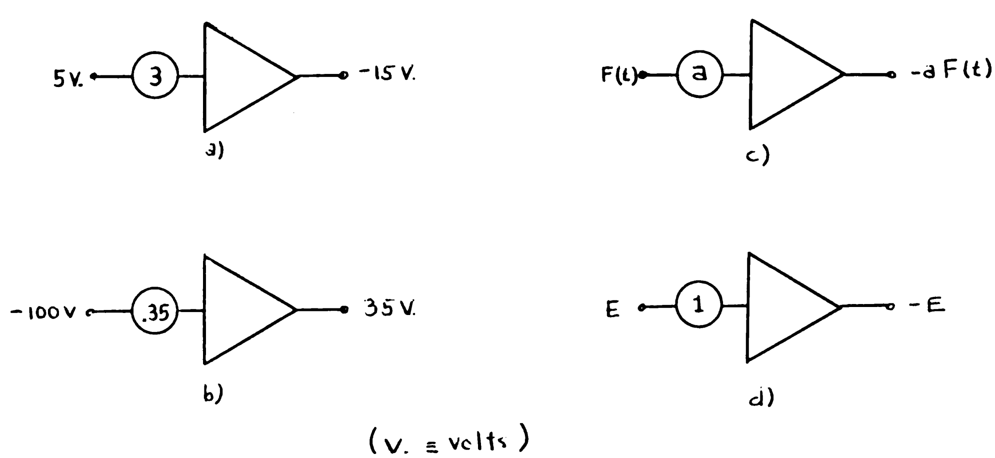


Fig. 1.3.3

voltage E. This last block occurs so often, that it has been given the special name of sign-changer.

The reader who does not have a strong background in electronic circuits should not be confused with words such as "amplifier," "gain," "resistance," etc. The use of a modern analog computer does not require such a background since we can set up our problems in the form of logical diagrams which are circuit independent. After some practice, anyone can learn to interconnect the resistances, capacitances, and amplifiers to implement the diagrams. The most important phase of the problem-solving process is the construction of the logic diagram. The actual wiring of the computer components is the job of a technician, and many installations may provide this service to the user.

One thing must be kept in mind, however, even when we are programming at the logic diagram level, and this is that all analog components have certain restrictions on their operating ranges which must be satisfied. For example, there are limits on the size of the resistances R_0 and R_1 which may be used in connection with a particular amplifier design. There are also limits on the minimum and maximum gains of a block and on the minimum and maximum value of the input and output voltages.

Common ranges of resistance, gain and voltage are $.01 \leq R \leq 10$ (megohms), $1/50 \leq \text{Gain} \leq 50$, and $-100 \leq V \leq +100$ volts. The reader should consult the computer manual for the actual ranges in the machine at his disposal.

Notice that the common values of resistance are given in units of megohms (a million ohms or 10^6 ohms). This makes the megohm a convenient unit for analog circuits. The reader will often see the ohmic value given in "megs," so that a sign-changer may show the values of $R_0 = R_1 = 1$ meg.

As seen in Fig. 1.3.2, the multiplying constant K has to be adjusted by selecting the proper ratio of two resistances R_0 and R_1. Whenever two such resistances are not available, a potentiometer may be used in order to adjust the gain to the proper value.

Example 1.3.1

A value of K = 2.56 is needed, but two resistances which provide this ratio are not available. Other gains such as .01, .1, 1, 2, 5, 10, etc., are available. The procedure is to divide K by one of the available gains greater than K such that a gain K' less than 1 will result. The gain K' may then be set in a potentiometer. Two ways of doing this are shown in Fig. 1.3.5

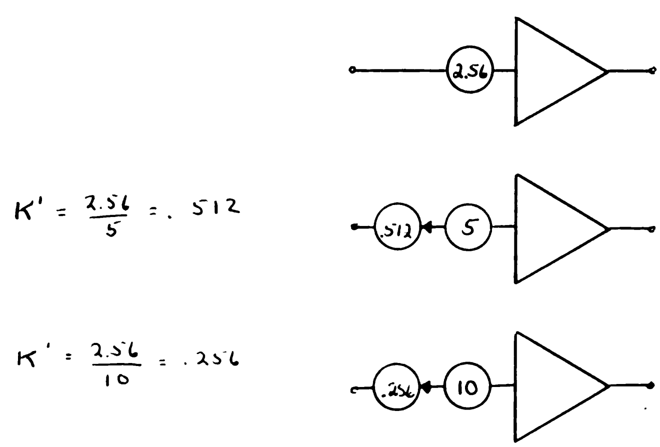


Fig. 1.3.5

1.4 Analog Addition

The addition of two machine voltages $E_1(t)$ and $E_2(t)$ is accomplished by the adder block or simply the adder or summer. The diagram of a two-input adder is given in Fig. 1.4.1a. The one-line diagram which will be used in these notes is given in 1.4.1b, and the block diagram of an n-input adder is shown in 1.4.1c.

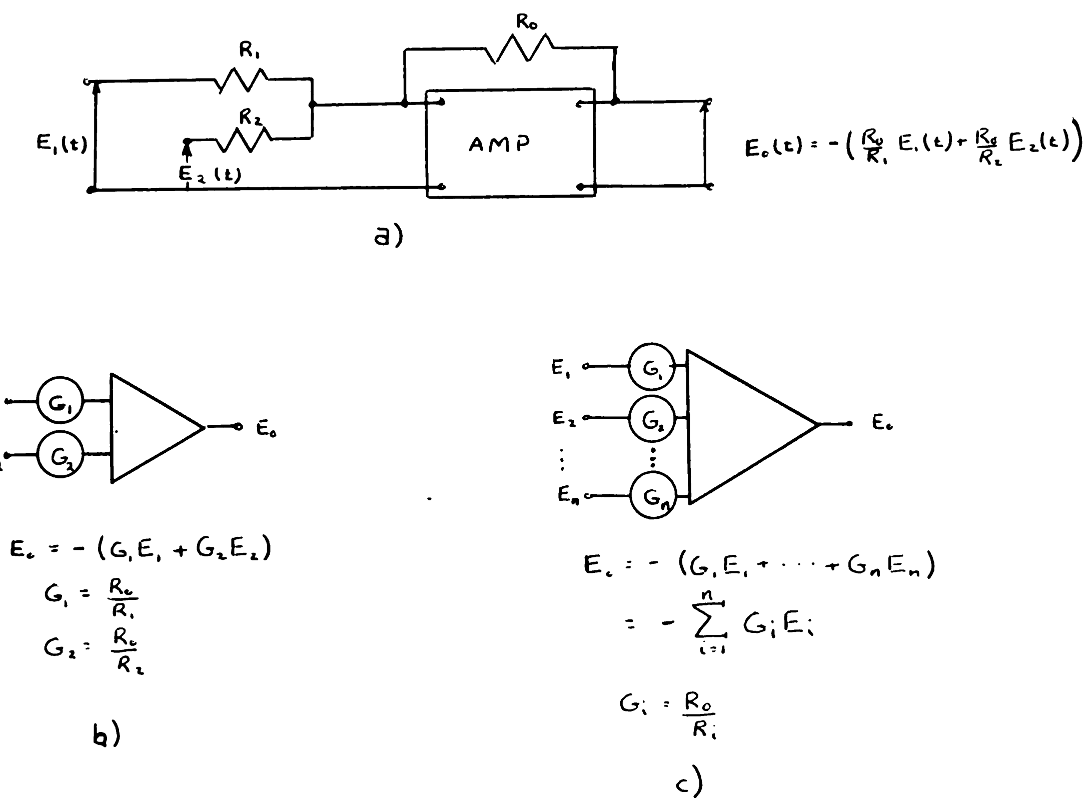


Fig. 1.4.1

Notice that the adder multiplies each input variable by its corresponding gain, adds these products and changes the sign of the sum. Fig. 1.4.2 shows two examples of the addition of analog variables.

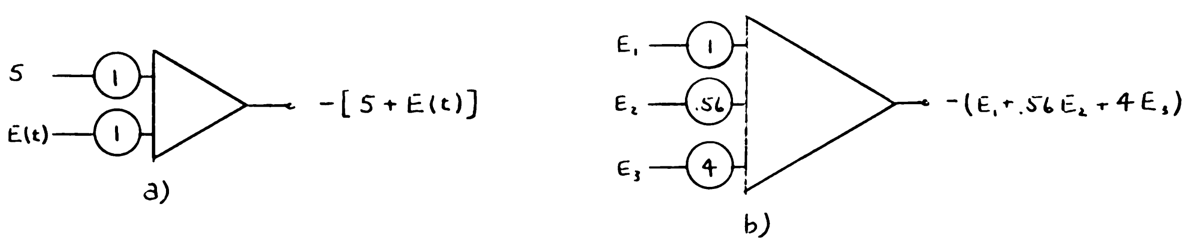


Fig. 1.4.2

Subtraction of analog variables may be done by changing the sign of the subtrahend with a sign-changer as shown in Fig. 1.4.3.

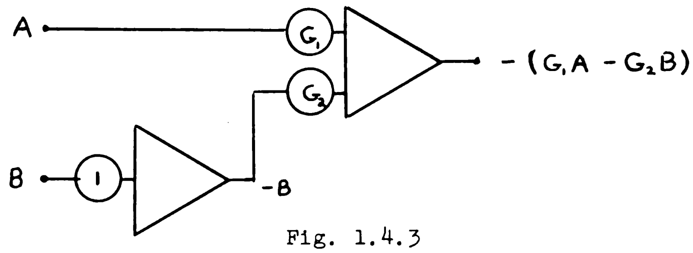


Fig. 1.4.3

1.5 Integration

Before we discuss an analog block which may be used for integration, it is worthwhile to review some basic concepts of integration. First, we remember that the symbol

$$\int_a^b f(x)dx \qquad (1.5.1)$$

is called the definite integral of f(x). The function f(x) is called the integrand and the numbers a,b are called limits of integration. Now, if f(x) is continuous in the interval $a \leq x \leq b$, and if f(x) is the derivative of another function called F(x),

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b)-F(a). \qquad (1.5.2)$$

Corresponding functions f(x) and F(x) may be found in tables of indefinite integrals of the form

$$F(x) = \int f(x)dx$$

Notice that the indefinite integral symbol does not have any limits of integration.

For example, if f(x) = cos(t) we recall from memory, or find in a table, that cos(t) is the derivative of sin(t). we may then write

$$\int_a^b \cos(t)dt = \sin(t)\Big|_a^b = \sin(b)-\sin(a) \qquad (1.5.3)$$

In analog computer work the most common integrals are time dependent, such as

$$\int_o^t f(t)dt = F(t)\Big|_o^t = F(t) - F(o). \qquad (1.5.4)$$

The F(o) term may be called the initial condition of the function represented by the indefinite integral.

Now we shall describe the analog block which may be used for the evaluation of integrals.

Integration of a machine voltage with respect to time is done with the integration block or integrator shown in Fig. 1.5.1a. The integrator consists of an operational amplifier, a resistance, and a capacitance. If the voltage across the capacitor is zero (the capacitor is discharged) when the input voltage E_1 is applied, the output voltage will be equal to a constant times the definite integral of the input voltage with respect to time. The constant is equal to the -1/RC ratio where R is in ohms and C is in farads, and is called the gain of the integrator. Fig. 1.5.1b shows the block diagram of the integrator. In order to specify that the capacitor is discharged when the integration process begins, an oval with a 0 in it is shown below the triangle.

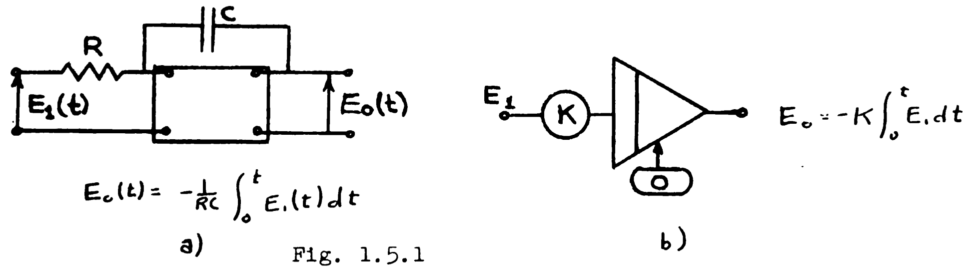


Fig. 1.5.1

For example: suppose that in Fig. 1.5.1 $E_1(t)$ = Sin(t) and k = 1. We obtain

$$E_o(t) = -\int_0^t \sin(t)dt = -\left[(-\cos(t))\Big|_0^t\right]$$
$$= -\left[-\cos(t) + \cos(0)\right] = \cos(t) - 1$$

If the capacitor is not discharged at the beginning of the integration process, the voltage across the capacitor will be superimposed as a constant on the output of the integrator. For example, if the voltage across the capacitor is 10 volts as shown in Fig. 1.5.2, the output voltage will be equal to cos(t)-1 + 10.

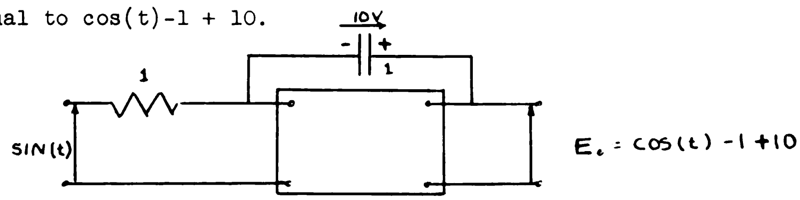


Fig. 1.5.2

On the other hand, if the voltage across the capacitor is reversed, as shown in Fig. 1.5.3, the output will be equal to cos(t)-1 - 10.

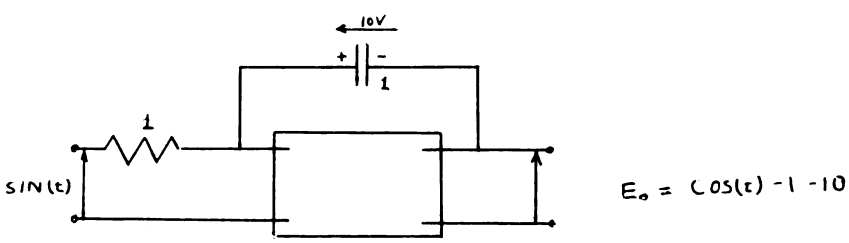


Fig. 1.5.3

The action of the capacitor voltage is shown symbolically in Fig. 1.5.4

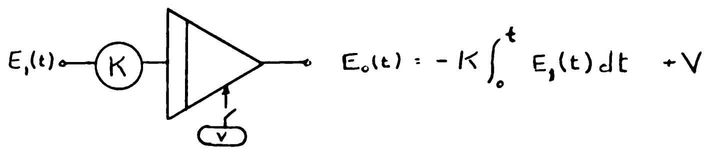


Fig. 1.5.4

In this figure the capacitor voltage is indicated inside the oval. This voltage is applied to the capacitor by means of a switch. At the beginning of the integration process (t = o), the switch is closed and the capacitor is charged to the voltage in the oval. This appears as a constant term added to the definite integral. As long as the switch is closed the integrator output voltage is held constant and equal to the voltage across the capacitor. For this reason the voltage in the oval is called the initial condition voltage. In some computers the initial condition voltage is applied to an amplifier input called the initial condition terminal. This initial condition input is similar to the other amplifier inputs except that a switch in series with it may be used to connect or disconnect this input from the amplifier. In this case the amplifier reverses the sign of the initial condition voltage so that a negative voltage -V will appear as a positive constant +V at the output, as shown in Fig. 1.5.5

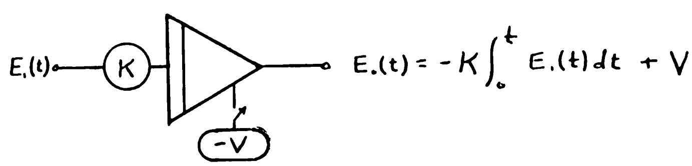


Fig. 1.5.5

Rather than using the convention of Fig. 1.5.5, which is computer oriented, in these notes we will write in the oval the actual initial condition value as in Fig. 1.5.4. Users of computers which produce a sign reversal should take this fact into account or may prefer to use the diagram of Fig. 1.5.5.

A fact that will help us later when we start using the integrator is that if the initial condition voltage is equal to the factor-F(o) of equation (1.5.4), then the integrator output is simply the indefinite integral of the input voltage. This is shown in Fig. 1.5.6

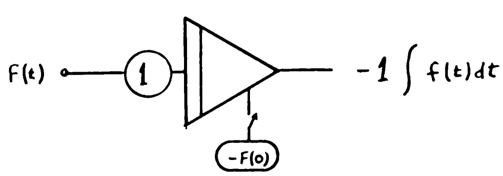
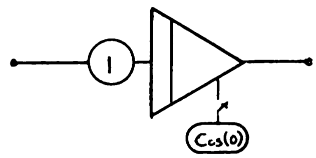


Fig. 1.5.6

Example 1.5.2

a) sin t



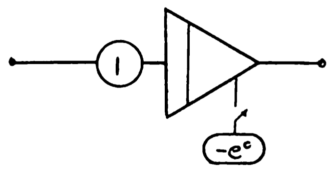
$$E_o(t) = -\int_o^t \sin t\, dt + \cos 0$$

$$= -(-\cos t)\Big|_o^t + \cos 0$$

$$= \cos t - \cos 0 + \cos 0$$

$$= \cos t = -\int \sin t\, dt$$

b) e^t



$$E_o(t) = -\int_o^t e^t dt \quad -e^o$$

$$= - e^t + e^o - e^o = -e^t$$

$$= -\int e^t dt$$

When using the integrator it must be remembered that the indefinite integral will appear as the output only if the initial condition voltage is adjusted to the value of F(o). Otherwise, the expression representing the output voltage must be found by evaluating the definite integral. Example 1.5.2 showed two cases in which the indefinite integral resulted because the initial condition voltages were set properly. If, on the other hand, the initial condition voltage in Example 1.5.2a is set equal to -10 volts, then the output is given by

$$E_o(t) = -\int_o^t \sin t\, dt - 10 = \cos t - 1 - 10$$

$$= \cos t - 11$$

The examples of Fig. 1.5.7 show what the output of an integrator will be when one machine unit (1 volt) of different polarities is applied to the input. Notice that only in the top example may we use the indefinite integral for the evaluation of the output.

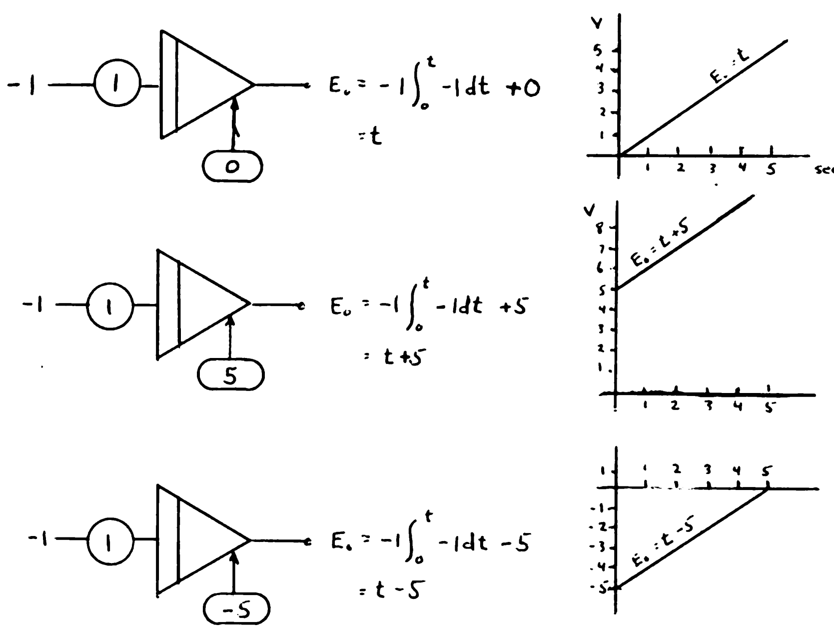
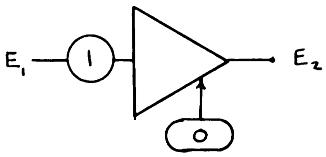


Fig. 1.5.7

Fig. 1.5.8 shows an example involving a time varying input voltage.



E_1 = -50 cos t
E_2 = 50 sin t

(real time in seconds and angle in degrees are analogous in this example)

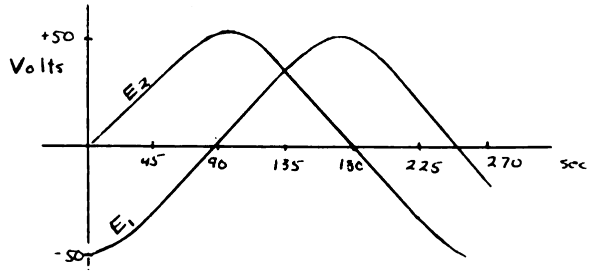
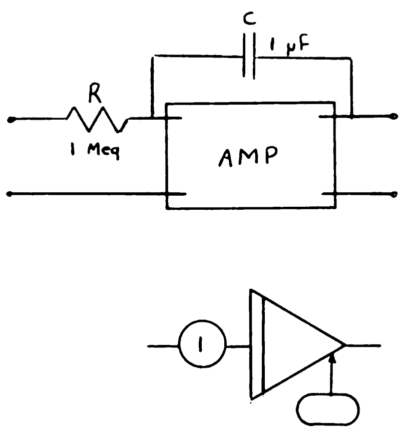


Fig. 1.5.8

In most computers the 1/RC ratio is adjusted by varying the resistance rather than the capacitance. This is done because it is easier to measure resistances accurately than capacitances. The most common capacitor used has a capacitance of 1 microfarad ($1x10^{-6}$ farads) so that a ratio of unity may be obtained by using a 1 megohm ($1x10^6$ ohms) resistor as seen in Fig. 1.5.9.



$$\frac{1}{RC} = \frac{1}{(1 \times 10^6 \text{ ohms})(1 \times 10^{-6} \text{ farads})} = 1$$

Fig. 1.5.9

Other ratios may be obtained by increasing or decreasing the value of R or by using potentiometers as was done in Example 1.3.1.

Example 1.5.1

Give the block diagram of an integrator with a gain of 1.58. Solution: If we assume that integer gains of 1, 2, and 5 are available, the diagrams of Fig. 1.5.10 are equivalent solutions.

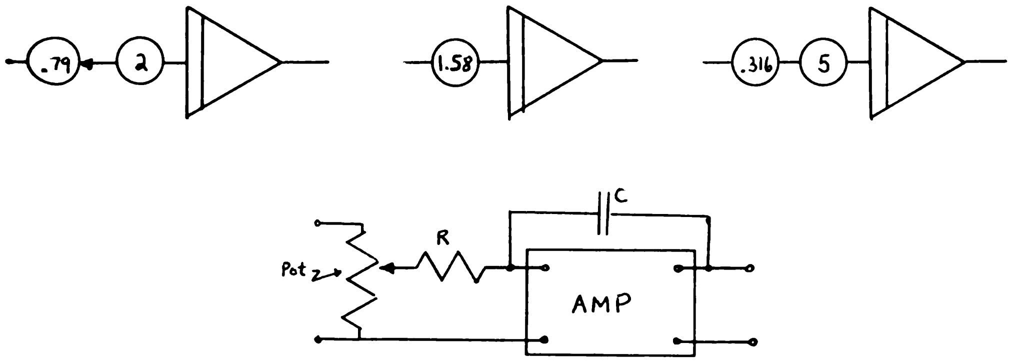
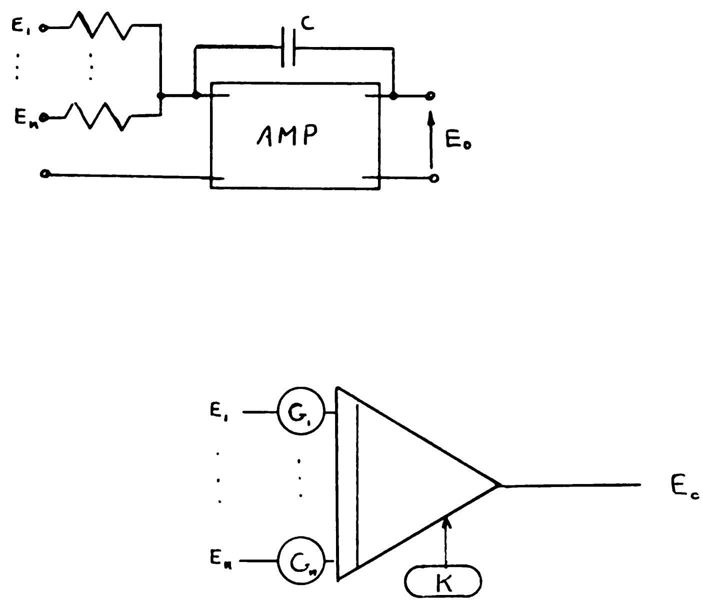


Fig. 1.5.10

1.6 The Summer-Integrator

The weighted sum of the integrals with respect to time of several time-varying input voltages may be obtained by the summer-integrator or adder-integrator in Fig. 1.6.1.



$$E_c = -\int_0^t (G_1E_1 + \cdots + G_nE_n)\,dt + K$$

Fig. 1.6.1

Examples of the use of this block are given in Fig. 1.6.2.

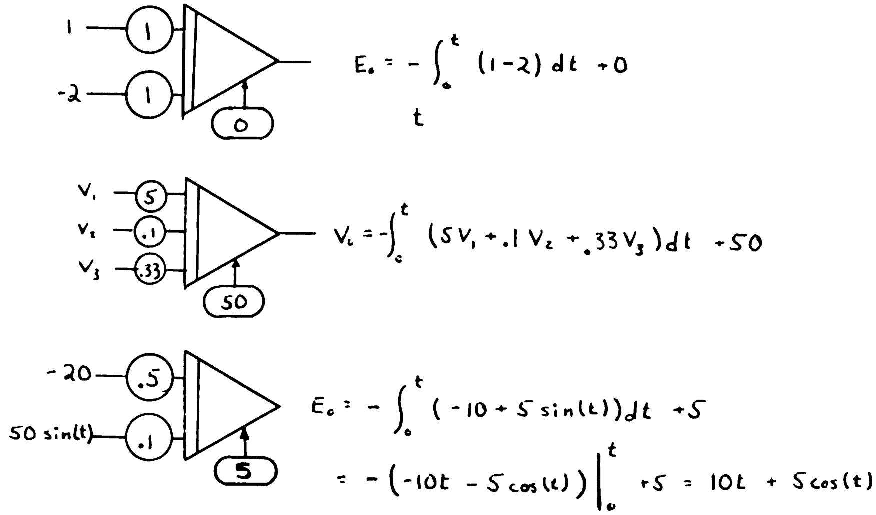


Fig. 1.6.2

1.7 Function Generators and Multipliers

In paragraph 1.2 we discussed the need for devices which generate arbitrary functions of one or more variables and devices which produce the instantaneous product of two or more variables which are changing as functions of time. Devices called function generators and function multipliers are used for these two operations.

Function Generators: Fig. 1.7.1a shows the general block diagram of a function generator. In its most general form this device accepts n time-varying voltages v_1, v_2,...,v_n and produces an output voltage which is a prescribed function of the input voltages.

In some cases a function generator may consist of a few electronic diodes and resistances. In general, however, a function generator is a complicated piece of equipment containing many components. In commercial models the function may be set or changed by various means, such as by setting dials, inserting templets with the same shape as the graph of the function, etc.

The most common types of function generators are those which produce a function of a single variable as shown in Fig. 1.7.1b and c. Fig. 1.7.1c results when time is the argument of the function, so that the generator needs no input voltage, since time may be supplied internally by integrating a constant voltage as was done in Fig. 1.5.7.

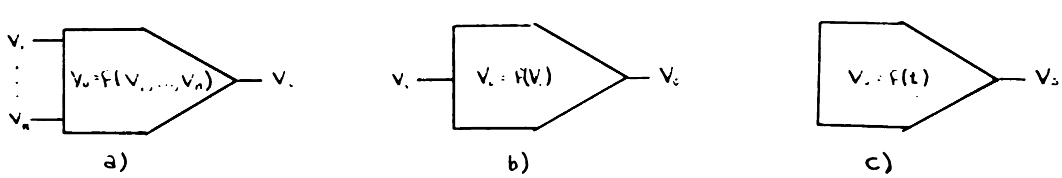


Fig. 1.7.1

Common examples of function generators of the type shown in Fig. 1.7.1c are the sine-wave, square-wave, and triangular wave oscillators available in any electronics laboratory.

Fig. 1.7.2 shows two examples of the generation of functions. The reader should study these examples carefully.

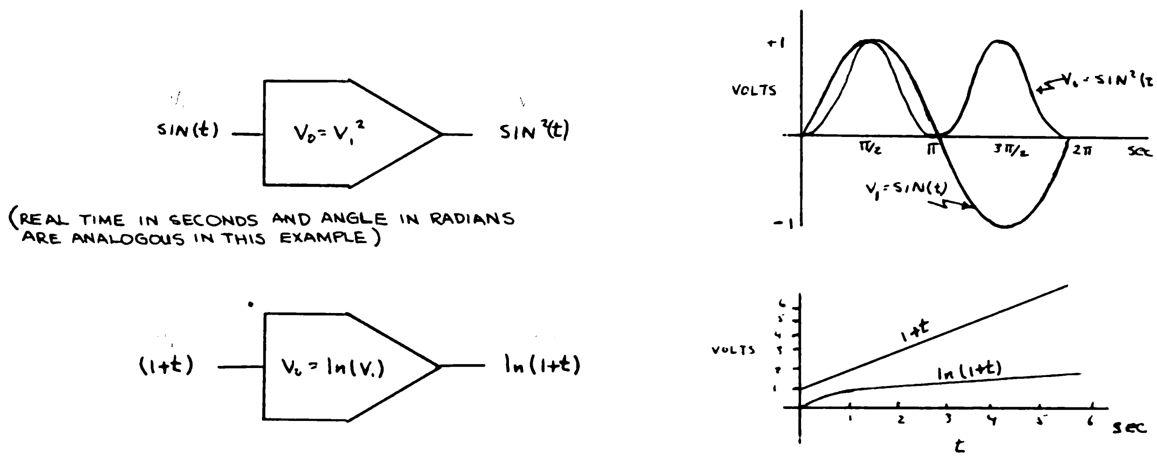


Fig. 1.7.2

<u>Function Multipliers</u>: The general block diagram of a function multiplier is shown in Fig. 1.7.3a. Each input variable is in general a continuous voltage changing with time. The multiplier develops an output voltage equal to a constant times the instantaneous product of the input voltages. The multiplying constant K may be either positive or negative. In these notes we will show the constant inside the block.

Fig. 1.7.3b shows a common type of function multiplier. This is a special case of the general multiplier which allows the multiplication of two variables only, and has a scale factor of -.01.

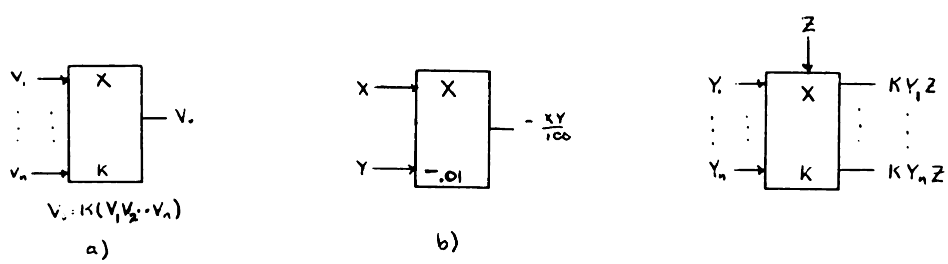


Fig. 1.7.3

Another common type of multiplier might be represented by the block diagram of figure 1.7.3c, in which a variable Z may be multiplied by each of n other variables $y_1, \ldots, y_n$. This multiplier, usually of the "servo" type, then provides n outputs which are proportional to the product of Z with each of the n inputs.

An example of the use of the two-input multiplier is given in Fig. 1.7.4. Two input voltages v_1 = 10sinθ and v_2 = 10θ are multiplied instantaneously to produce an output equal to -.01 (10 sin θ)(10 θ) = -θ sin θ.

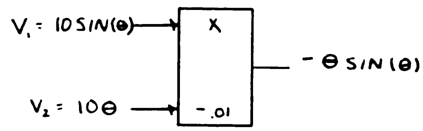


Fig. 1.7.4

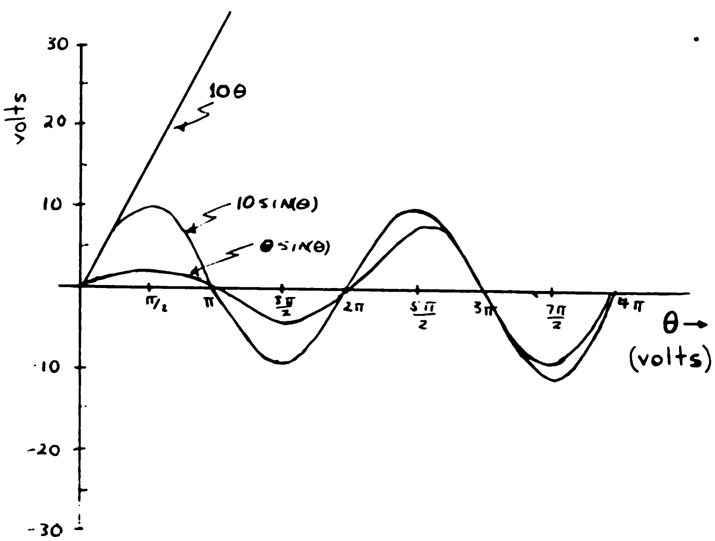


Fig. 1.7.5

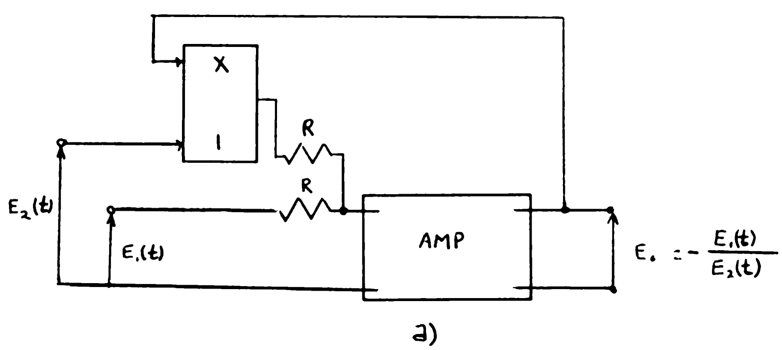
Notice that in the function v_1 = 10sinθ the analogy between radians and volts is implied. When the angle θ is equal to π/2 radians its analog is equal to 1.5708 volts and the sine of the angle is equal to 1, so that the voltage at the v_1 terminal would be 10 volts. If an ordinary d-c voltmeter is connected between ground and this terminal, it would read 10 volts at the time when θ is 1.5708 volts.

Many methods have been developed for obtaining function multiplication. Perhaps the most popular methods are provided by the servo-multiplier and the square-law electronic multiplier. The most common values for the constant K in Fig. 1.7.1 are ± 1 and ± .01, depending on the particular multiplier. The accuracy with which multiplication is performed depends on the frequency of the input voltages. Static accuracies of .1% are common and accuracies of .01% or better may be obtained in more expensive equipment. The dynamic accuracy deteriorates as the frequency increases, and it is better in the electronic multiplier than in the servo-multiplier. A detailed explanation of the different methods for obtaining function multiplication is not within the scope of these notes.

The reader might notice that the function generator is the most general analog block, since all of the blocks discussed are special cases of it. For example, an integrator is a function generator which accepts a voltage and generates its time integral, an adder accepts several input voltages and develops their sum, a multiplier is similar to an adder except that it develops the product of the input variables, etc. As a matter of fact, we showed in Fig. 1.5.7 how the functions t, t+5, t-5 may be generated with an integrator.

A Multiplier used for Division

A function multiplier may be used for division by connecting it to an amplifier as shown in Fig. 1.7.6a. Notice that the amplifier serves as an adder whose output voltage is fed back to the multiplier. Fig. 1.7.6b gives the block diagram of a divider.



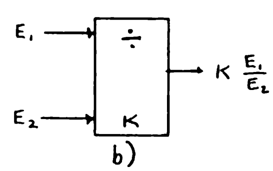


Fig. 1.7.6

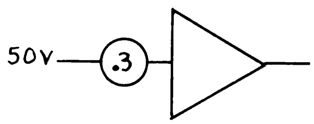
1.8 Other Analog Blocks

There are many analog computer components which we have not mentioned here. The blocks which have been presented, however, are enough for solving a large percentage of the problems which are tackled by analog techniques.

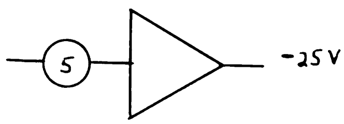
The output devices, that is, meters, recorders, etc. which are needed to obtain the final answers are explained in Chapter 3, where a sample computer is described.

Exercise 1.1

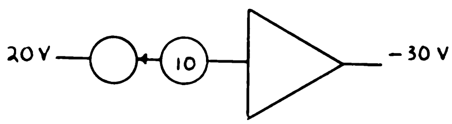
DRILL EXERCISE ON ANALOG BLOCKS:

a. 

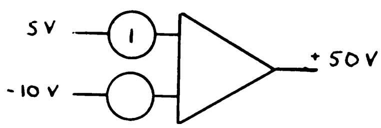
output voltage =

b. 

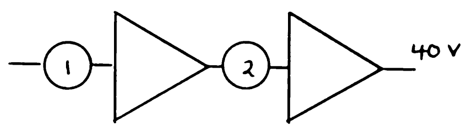
input voltage =

c. 

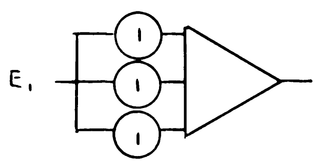
pot setting =

d. 

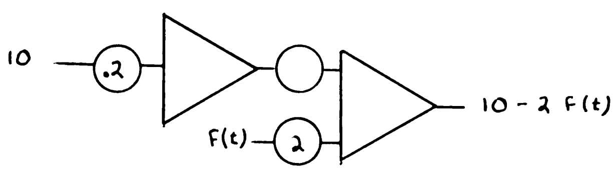
missing gain =

e. 

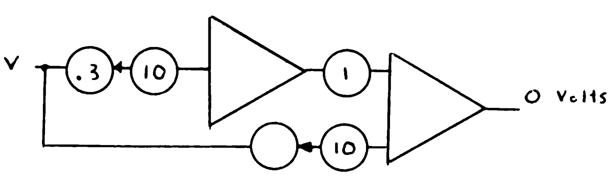
input voltage =

f. 

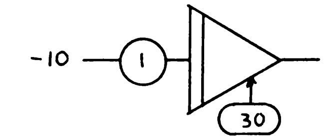
output voltage =

g. 

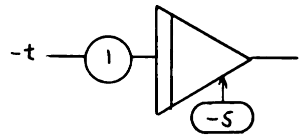
missing gain =

h. 

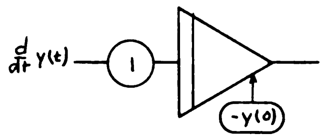
pot setting =

i. 

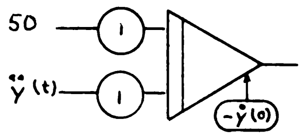
output voltage (as a function of time) =

j. 

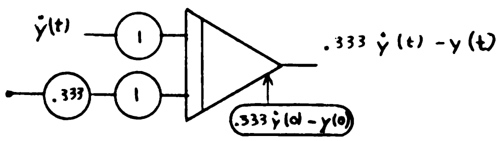
output voltage =

k. 

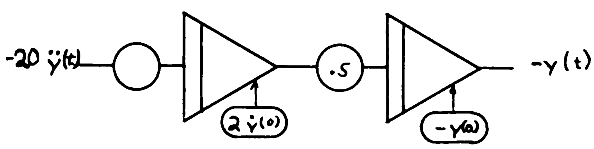
output voltage =

l. 

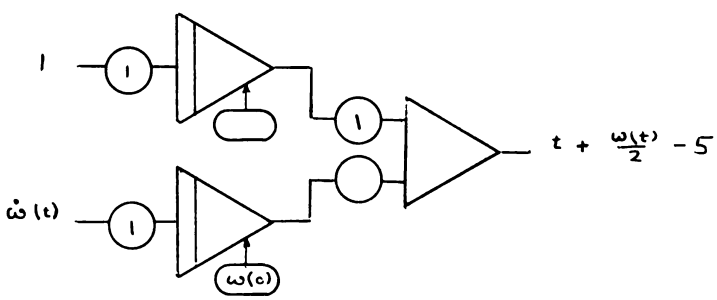
output voltage =

m. 

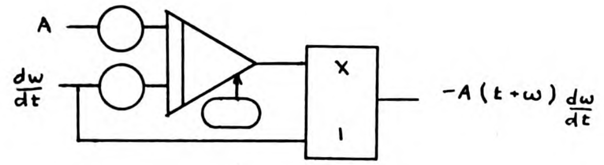
missing input =

n. 

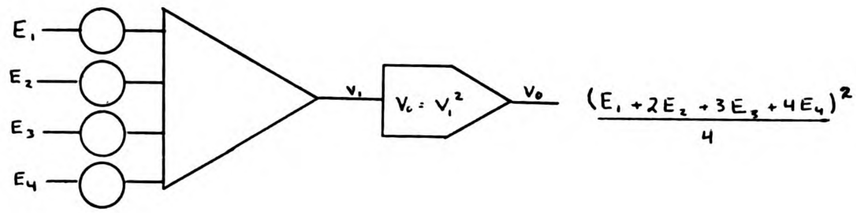
missing gain =

o. 

missing initial condition =
missing gain =

p. 

missing gains and initial condition =

q. 

missing gains =

PROBLEMS

1.1 Generate the function $-\frac{1}{2}(\dot{\theta} + \dot{\beta}) + \dot{\alpha}$ from the functions $\ddot{\theta}$, $\ddot{\beta}$, and $-\ddot{\alpha}$ by using a single analog block.

1.2 Notice that

$$\left[\frac{y+z}{2}\right]^2 - \left[\frac{y-z}{2}\right]^2 = yz.$$

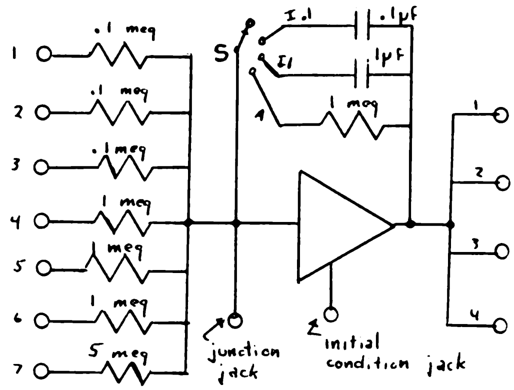
Use this idea to generate function multiplication from two identical function generators which generate the squaring function plus any additional analog blocks needed to form the terms to be squared.

1.3 Draw the block diagram of an analog circuit for generating the function $-t \cdot \ln(B)$, assuming that the function $v_o = \ln(v_1)$ may be implemented with a function generator which accepts the voltage v_1 and produces the voltage v_o.

1.4 Draw the block diagram of an analog circuit which will accept 5 variables $x_1, \ldots, x_5$ and generate $\sum_{i=1}^{5} x_i^2 - \frac{\left(\sum_{i=1}^{5} x_i\right)^2}{5}$. Generalize the block diagram to accept n variables.

1.5 A certain analog computer has operational amplifiers which are pre-wired as shown in the drawing. The amplifier has a selector switch S which converts it into an adder if the switch is in position A, a summing integrator if in positions I1 or I.1. The following jacks are available for connections: the seven input jacks, four common output jacks, a junction jack, and an initial condition jack. Inputs which are not needed may be left disconnected. Thus, this amplifier may be used as a universal analog block. Constant 1 megohm resistors are available and may be connected to the junction jack when more inputs are needed.

1.5 (cont'd)



Show how the following blocks may be implemented:

a. sign changer

b. constant multiplier (How many different constants can you get without the use of attenuators?).

c. three input adder with gains 1,2,5 (Hint: resistances may be paralleled or attenuators may be used).

d. an integrator with unity gain.

e. a summing integrator with gains 1,.5,2

Chapter 2 ANALOG SOLUTION OF EQUATIONS

2.1 Introduction

In Chapter 1 we discussed the basic operations which may be performed on voltages by the components of analog computer Now we shall show how an analogy may be set up between the voltages in the computer and the variables in certain types of problems.

In the discussion which follows we shall prefer to use the block diagram of the analog components rather than the circuit diagram. By doing this, the diagrams will be useful regardless of the type of analog computer on which the problem is solved.

2.2 Linear Equations

The simplest equation that may be solved in an analog computer is a linear equation. For example, the equation

$$y = mx + b, \tag{2.2.1}$$

which is recognized as the equation of a straight line with intercept b and slope m, may be generated by means of the analog block in Fig. 2.2.1.

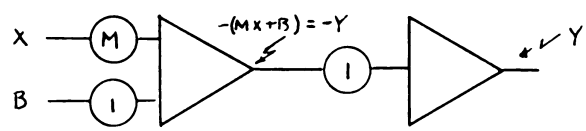


Fig. 2.2.1

The equation may be solved by assuming that two voltages X and B are analogous to the mathematical variables x and b. If these voltages are fed to the two inputs of an adder as shown in Fig. 2.2.1 and the gains are adjusted as shown in the figure, the output of the adder will be equal to the negative of the sum of the inputs times their repective gains, which according to the given linear equation, is equal to -Y. The output of the sign changer is then a voltage analogous to the mathematical variable y. The voltages X, B, and Y are the machine variables, and they are analogous to the problem variables x, b, and y.

Other computer connections which yield the solution of (2.2.1) are given in Fig. 2.2.2. The reader should study the diagrams and prove to himself that they only differ from Fig. 2.2.1 in the way in which the machine variables are defined and distributed. For example, in 2.2.2a the problem variable b is made analogous to the gain B rather than to a voltage, and in 2.2.2d negative voltages are used for X and B in order to eliminate the sign changer. This is easily done, since analog computers have a voltage supply which provides both polarities of voltage.

Preference for one of the diagrams in Fig. 2.2.2 depends on whether it is easier to change the magnitude of a voltage or the magnitude of a gain. This depends on the computer which is being used.

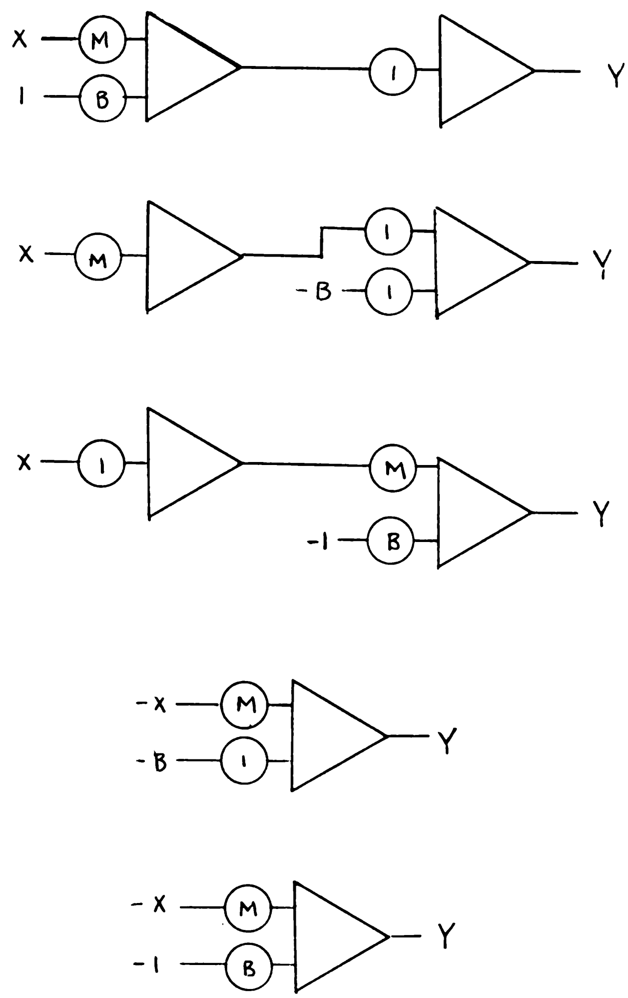


Fig. 2.2.2

The problem variables x, y, m, and b are given in units which depend on the physical problem for which the equation is a mathematical model. For example, one form of the linear equation which we remember from elementary physics is

$$y = vt + y_o \qquad (2.2.2)$$

which gives the total distance y traveled during the time t by an object with a constant velocity v and an initial distance y_o from a reference point. If y and y_o are in miles, v in miles per hour and t in hours, the analogy between the physical and problem variables in Fig. 2.2.3 will be:

voltage T is analogous to hrs. of time t

voltage Y_o is analogous to miles y_o

voltage Y is analogous to miles y

non-dimensional gain v is analogous to miles/hr v

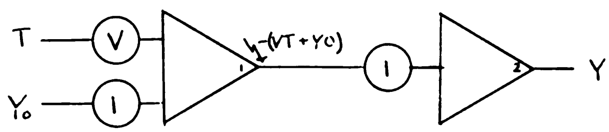


Fig. 2.2.3

Now we may ask: What voltages T, Y, and Y_o are equivalent to hrs. of t, and miles of y and y_o? Obviously, we must set up proportionality constants between each machine variable and each problem variable so that the machine voltages will always be within the range for which the analog blocks were designed. If for a particular problem t is equal to 2000 hrs, it is obvious that a one-to-one correspondence between t and T might damage an analog block which is designed to operate at a maximum voltage of $\pm$ 100 volts, since this would necessitate feeding 2000 volts to one of the inputs of amplifier 1 in Fig. 2.2.3.

Let us investigate in more detail how the relationships between problem variables and machine variables may be found. We will do this with a simple example.

Example 2.2.1

Given: $v = 2$ miles/ hour

$y_o = 100$ miles

Find: y for $0 \leq t \leq 200$ hrs.

Solution: Equation (2.2.2) becomes

$$y = 2t + 100 \quad (2.2.3)$$

The maximum value of y is

$$Y_{max} = 2\ t_{max} + 100$$
$$= 2 \times 200 + 100 = 500 \text{ miles} \quad (2.2.4)$$

If this equation is to be solved by analog techniques, the machine variables Y, T, and Y_o must never exceed the maximum operation voltages or gains of the analog blocks. If a one-to-one correspondence is set up between machine and problem variables, and if the diagram of Fig. 2.2.3 is used, the output of amplifier 2 would be required to deliver 500 volts and one of the inputs of amplifier 1 (T) would require a maximum voltage of 200 volts. Assuming that our computer does not handle voltages above ± 100 volts, we must use a set of appropriate scale factors rather than a one-to-one relationship between problem and machine variables. For example, since y reaches a maximum value of 500, each machine unit of Y (volts) may be made equivalent to 5 problem units of y (miles), each machine unit of T (volts) equivalent to 2 problem units of t (hrs.) This is written

$$\begin{cases} 2T = t, \\ 5Y = y, \end{cases} \qquad (2.2.5)$$

and corresponds to the process of changing variables in mathematics.
When these relationships are substituted into (2.2.3) we obtain

$$5Y = 2\cdot(2T) + 100$$

and

$$0 \leqslant 2T \leqslant 200, \qquad (2.2.6)$$

which after simplification, yield the machine equations

$$\begin{cases} Y = \frac{4}{5}\cdot T + 20 \\ 0 \leqslant T \leqslant 100, \end{cases} \qquad (2.2.7)$$

and the circuit given in Fig. 2.2.4.

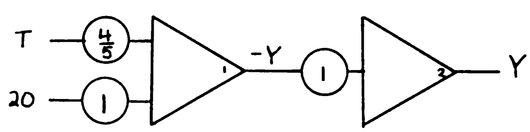


Fig. 2.2.4

We must keep in mind that in order to interpret the analog results, the machine units of volts must be changed into problem units by applying the scale factors in reverse. For example, when T = 50 volts, the output of amplifier 2 in Fig. 2.2.4 will be given by (2.2.7) and it will be equal to 60 volts. The machine variables are given by

$$t = 2T = 2\times 50 = 100\text{hrs.}$$

$$y = 5Y = 5 \times 60 = 300 \text{ miles}$$

Another way to interpret the change in scale is to draw the block diagram shown in Fig. 2.2.6. Here the relationships of (2.2.5) are used to label the block diagram in terms of the original problem variables.

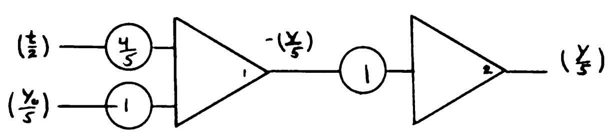


Fig. 2.2.6

Notice that in this diagram the actual problem variables t, y, and y_o are shown rather than the scaled machine variables T, Y, and Y_o. When this approach is followed, the analog circuit is adjusted in such a way that the outputs of the blocks will yield fractions of the problem variables rather than the variables themselves. In this way we avoid the overloading of the individual blocks.

Notice, for example, that amplifier No. 1 would yield

$$-\frac{y}{5} = -\left(\frac{4}{5} \cdot \frac{t}{2} + \frac{y_o}{5}\right) \qquad (2.2.8)$$

or

$$y = 2t + y_o \qquad (2.2.9)$$

which is indeed the given linear equation. When $y = y_{max} = 500$ miles, the output of amplifier 1 will be y/5 = 500/5 = 100 volts. If this output is recorded, the recorded result should be labeled y/5 and interpreted accordingly.

The diagram of Fig. 2.2.6 may be obtained by multiplying and dividing each variable to be scaled by the same scale factor so that the equation is not affected. In the case of equation (2.2.3) we would like to use a scale factor of 5 for y and a factor of 2 for t, so that the equation may be written

$$5\left(\frac{y}{5}\right) = 2 \cdot 2\left(\frac{t}{2}\right) + 100 \qquad (2.2.10)$$

The $\left(\frac{y}{5}\right)$ term is then isolated to give

$$\left(\frac{y}{5}\right) = \frac{4}{5}\left(\frac{t}{2}\right) + 20, \qquad (2.2.11)$$

which is the equation represented by Fig. 2.2.6.

Another example will further illustrate this procedure.

Example 2.2.2

Draw the block diagram of an analog circuit for finding the value of P as a function of Q and R given by the equation

$$P = 400Q + .02R - 500, \qquad (2.2.12)$$

for

$$0 \leq Q \leq 2 \qquad (2.2.13)$$

$$0 \leq R \leq 10^4 \qquad (2.2.14)$$

1) The maximum values of Q and R are specified by (2.2.13) and (2.2.14). The maximum and minimum values of P are be found to be

$$P_{min} = 400 \cdot 0 + .02 \cdot 0 - 500 = -500$$

$$P_{max} = 400 \cdot 2 + .02 \times 10^4 - 500 = +500$$

2) The three variables must be scaled. This may be done in many different ways. One possibility is to draw an analog circuit which will display the variables 50Q, R/100, and P/5. If this is done, the maximum voltages represented by these variables will never exceed ±100 volts. The equation is rewritten in the form

$$5\left(\frac{P}{5}\right) = 400\cdot \frac{1}{50} (50Q) + .02\cdot 10^2\left(\frac{R}{10^2}\right) - 500,$$

which yields after simplification

$$\left(\frac{P}{5}\right) = \frac{8}{5}\cdot (50Q) + \frac{2}{5}\cdot\left(\frac{R}{10^2}\right) -100 \qquad (2.2.15)$$

3) The diagram of Fig. 2.2.7 is then drawn.

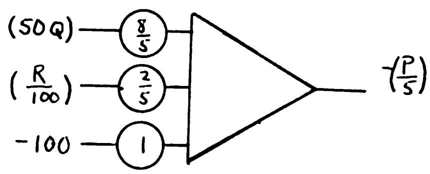


Fig. 2.2.7

Notice that the maximum positive value of P is obtained when Q = 2 and R = 10^4. In terms of our scaled diagram, this occurs when (50Q) = 100 volts and $\frac{R}{100}$ = 100 volts. The diagram of Fig. 2.2.8 shows this condition.

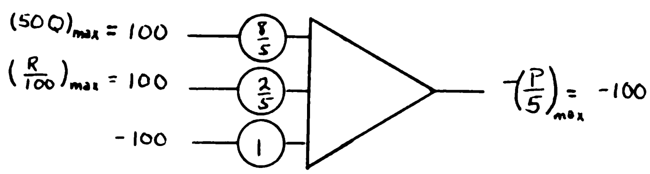


Fig. 2.2.8

<u>Another Solution</u>:

Suppose that for some reason we wish to display 40Q, $\frac{R}{200}$, and $\frac{P}{10}$. In this case the equations written in terms of these scaled variables become

$$10\left(\frac{P}{10}\right) = 400\cdot \frac{1}{40} (40Q) + .02\cdot 200\left(\frac{R}{200}\right) -500,$$

$$\left(\frac{P}{10}\right) = (40Q) + .4\cdot\left(\frac{R}{200}\right) -50,$$

The diagram is shown in Fig. 2.2.9.

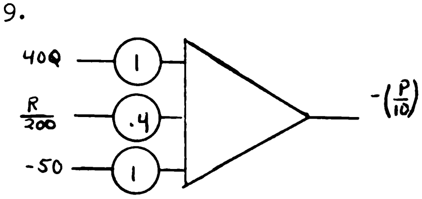


Fig. 2.2.9

The diagram shown in Fig. 2.2.9 will display a maximum voltage of -50 volts when Q and R reach the given maximum values.

2.3 Solution of Differential Equations

The solution of differential equations with the analog computer will be explained with an example of a simple ordinary differential equation with constant coefficients.

Consider the differential equation

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = 0 \qquad (2.3.1)$$

and the initial conditions

$$y(0) = y_{00} \qquad (2.3.2)$$

$$\left.\frac{dy}{dt}\right|_{t=0} = y_{10} \qquad (2.3.3)$$

These equations may be written in "dot" notation as follows:

$$a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = 0 \qquad (2.3.4)$$

$$y(0) = y_{00} \qquad (2.3.5)$$

$$\dot{y}(0) = y_{10}. \qquad (2.3.6)$$

The first step in solving the equation is to set up an analogy between the problem variables and the machine variables or voltages. This is done by assuming that in some way we will be able to obtain a computer voltage which will change with time in the same fashion as y(t) changes with respect to time t. In the same manner, we will assume that voltages proportional to the first and second time derivatives of y are available in the computer. The second step is to draw an analog diagram which will relate these assumed voltages in such a way that equation (2.3.1) and its initial conditions are satisfied.

One of the most useful methods of finding the analog circuits of differential equations is the "highest derivative method". The method consists of solving for the highest derivative, in this case $\ddot{y}$ (t), to obtain

$$\ddot{y}(t) = -\left[\frac{a_1}{a_2} \dot{y}(t) + \frac{a_0}{a_2} y(t)\right] , \qquad (2.3.7)$$

and to assume that a machine voltage analogous to this derivative appears as the output of an adder which satisfies equation (2.3.7) as shown in Fig. 2.3.1

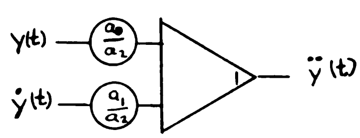


Fig. 2.3.1

We then realize that voltages proportional to $\dot{y}(t)$ and $y(t)$ are needed in order to satisfy the inputs of the adder. These voltages are obtained by repeated integration of the highest derivative as shown in Fig. 2.3.2

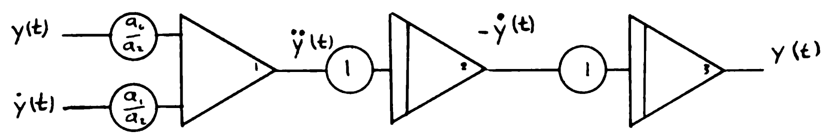


Fig. 2.3.2

Due to the change of sign produced by the analog blocks, the first derivative $\dot{y}(t)$ appears with a negative sign.

A sign-changer may be used as shown in Fig. 2.3.3 to generate the input needed by amplifier 1. This figure also shows that since the outputs of amplifiers 3 and 4 represent the desired inputs to amplifier 1, actual physical connections are provided in order to complete the circuit. Fig. 2.3.3 also shows how the initial conditions of $\dot{y}(t)$ and $y(t)$ are applied in order to complete the solution of the problem.

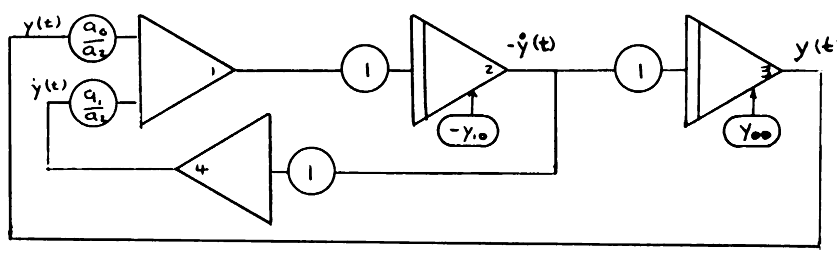


Fig. 2.3.3

Notice that the initial condition of amplifier 2 is not the initial value of $\dot{y}(t)$, but rather the initial value of $-\dot{y}(t)$, which is equal to $-y_{10}$.

Operating Procedure

After the logical diagram of Fig. 2.3.3 is drawn, the solution of the differential equation on the analog computer is obtained as follows:

1) The gains of the analog blocks are adjusted by choosing resistors and capacitors of the proper values or by using attenuators.

2) The blocks are interconnected by means of pluggable wires.

3) The initial condition voltages $-y_{10}$ and y_{00} are adjusted and applied to the initial condition terminals of the integrators. If the signs of these voltages are reversed by the amplifiers, this must be taken into account.

4) The initial condition switches are closed momentarily to charge the integrating capacitors. When this is done, the outputs of the blocks are

$$\text{Output of amplifier 1} = -\left(\frac{a_1}{a_2}\, y_{10} + \frac{a_0}{a_2}\, y_{00}\right)$$

$$\text{Output of amplifier 2} = -\, y_{10}$$

$$\text{Output of amplifier 3} = y_{00}$$

5) The initial condition switches are opened, permitting the integrators to start operating as such. The instant of time when the switches are opened is analogous to $t = 0$. The voltages $y(t)$, $-\dot{y}(t)$, and $\ddot{y}(t)$ constitute the solution to the problem, and may be recorded on a voltage recording device such as a strip-chart recorder.

The analog computer yields not only the dependent variable $y(t)$, but also its derivatives. The solution may be repeated as many times as desired by returning to Step 4. When this is done, it is possible to make changes in the parameters and the initial conditions by merely changing the gain of the blocks or the values of the initial condition voltages. A complete set of solutions may be run in a few minutes, since changing gains or voltage magnitudes is as easy as turning an attenuator dial.

2.4 The Non-Uniqueness of the Analog Circuits

If instead of solving for the highest derivative of (2.3.4) we would have chosen to solve for $a_2\ddot{y}(t)$, the equation

$$a_2\ddot{y}(t) = -\, a_1\dot{y}(t) - a_0 y(t)$$

would have resulted. It may be easily verified by following the same line of reasoning used in the previous section that this equation is solved by the diagram shown in Fig. 2.4.1.

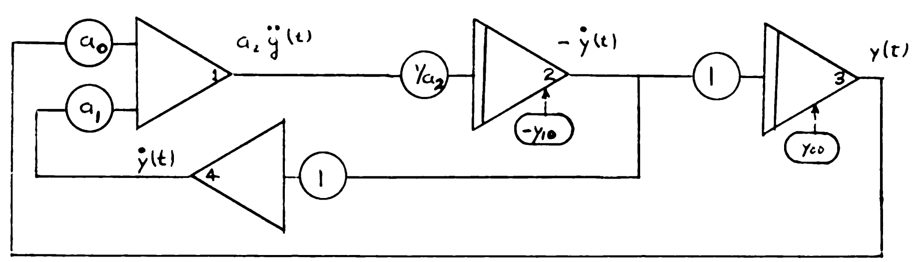


Fig. 2.4.1

Other diagrams that may be used are given in Fig. 2.4.2. The reader should study these diagrams carefully and try to find other variations.

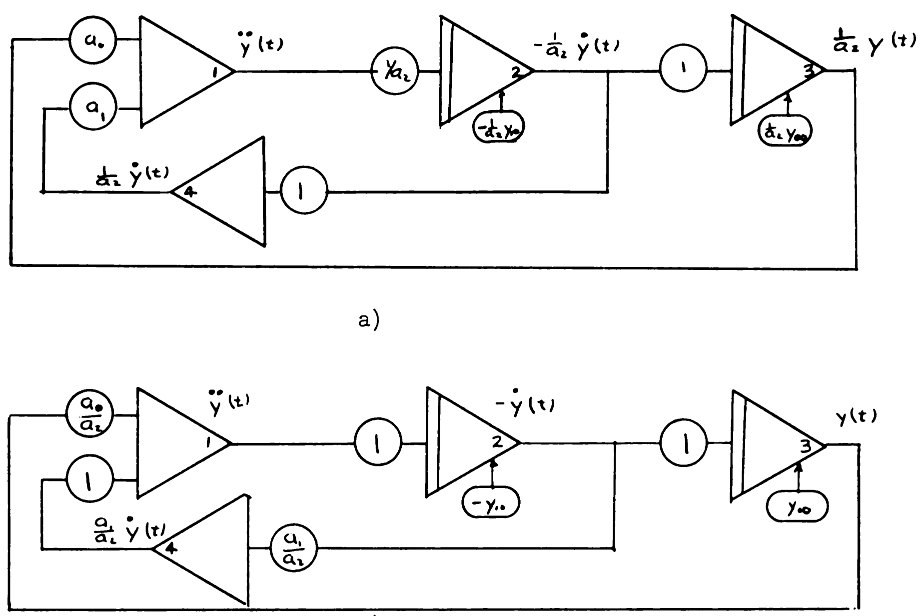


Fig. 2.4.2

The diagrams of Figs. 2.4.1 and 2.4.2 point out the fact that analog computer circuits are not unique, and that any one equation may be solved by many different circuits. A particular solution, however, may be more convenient in a particular case. For example, in Fig. 2.4.2a the gains of amplifier 1 are independent of the coefficient a_2. In some particular case this circuit may be more desirable than any other of the many possible circuits.

2.5 Scaling a Differential Equation

In section 2.2 we saw how a linear equation was scaled by choosing a proper set of display variables which guaranteed that the outputs of the amplifiers would operate within their useful range.

Scaling a differential equation is similar to scaling a linear equation, except that two new problems have to be solved. The first problem is that in differential equations it is sometimes necessary to change the scale of the independent variable, that is, to change the time scale, in order to operate within the frequency range of the analog devices. This problem is treated in paragraph 2.6. The second problem is that it is more difficult to estimate the maximum values of the variables in a differential equation than in a linear equation. This problem is treated in more detail in paragraph 2.7.

We shall first treat the problem of magnitude scaling, that is, the problem of keeping the magnitude of all the analog voltages within certain limits. The idea behind magnitude scaling is again to find a set of display variables which will make use of the full range of operation of the amplifiers and other analog components. The form of the display variables again depends on the maximum values of the original problem variables. We shall illustrate the scaling problem by giving an example.

Example 2.5.1

The equation

$$\ddot{x}(t) + 2\dot{x}(t) + 5x(t) = 0 \tag{2.5.1}$$

is to be solved with the initial conditions

$$x(0) = 20 \text{ In}$$
$$\dot{x}(0) = 20 \text{ In/sec.} \tag{2.5.2}$$

Assume that from the knowledge of the physical system the maximum values of the variables have been estimated to be

$$|x(t)|\ \max = 20 \text{ in}$$
$$|\dot{x}(t)|\ \max = 200 \text{ in/sec} \tag{2.5.3}$$
$$|\ddot{x}(t)|\ \max = 300 \text{ in/sec}^2$$

The analogy is now set up between a set of voltages in the computer and the problem variables $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$ such that voltage $x(t)$ in the computer is equivalent to inches $x(t)$ in the problem, voltage $\dot{x}(t)$ in the computer is equivalent to inches per second $\dot{x}(t)$ in the problem, etc. If this is the case, however, we will not be able to display $\dot{x}(t)$ and $\ddot{x}(t)$, since their maximum values would exceed $\pm$ 100 volts. At the same time the amplifier supplying $x(t)$ will only reach a maximum value of 20 volts, so that its full output range will not be utilized. The solution is to decide on a set of display variables which will make the maximum value of each voltage approximately equal to $\pm$ 100 volts. Such a set of display variables is:

$$\frac{100}{20}\,x(t),\ \frac{100}{200}\,\dot{x}(t),\ \text{and}\ \frac{100}{300}\,\ddot{x}(t)$$

or

$$5\,x(t),\ \frac{1}{2}\,\dot{x}(t),\ \text{and}\ \frac{1}{3}\,\ddot{x}(t).$$

The differential equation and its initial conditions are then written in terms of the display variables as follows:

$$3\left(\frac{\ddot{x}(t)}{3}\right) + 2 \cdot 2\left(\frac{\dot{x}(t)}{2}\right) + 5 \cdot \frac{1}{5}\,(5\,x(t)) = 0, \tag{2.5.4}$$

$$\frac{1}{5}\,(5\,x\,(0)) = 20, \tag{2.5.5}$$

$$2\,\left(\frac{\dot{x}(0)}{2}\right) = 20. \tag{2.5.6}$$

Simplifying and isolating the highest derivative we obtain

$$\left(\frac{\ddot{x}(t)}{3}\right) + \frac{4}{3}\left(\frac{\dot{x}(t)}{2}\right) + \frac{1}{3}\,(5\,x(t)) = 0, \tag{2.5.7}$$

$$(5\,x(0)) = 100, \tag{2.5.8}$$

$$\left(\frac{\dot{x}(0)}{2}\right) = 10. \tag{2.5.9}$$

The scaled highest derivative ($\ddot{x}(t)/3$) is assumed to exist at the output of amplifier 1 of Fig. 2.5.1. This output is integrated to obtain the next lower scaled derivative ($\dot{x}(t)/2$) which will appear at the output of amplifier 2. Another integration will produce the scaled solution (5 x(t)).

Notice that the gains of the integrators are adjusted so that their outputs and inputs will correspond to the scaled variables.

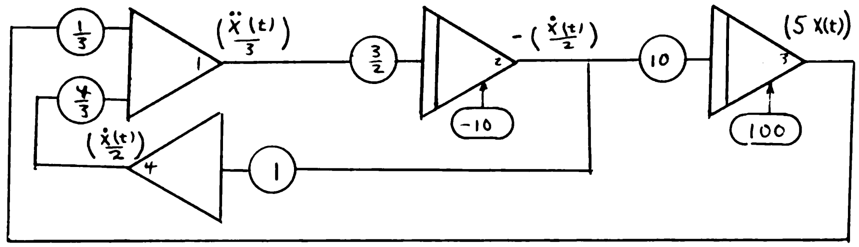


Fig. 2.5.1

The reader should convince himself that the gains of the integrators should be 3/2 and 10 respectively.

It is of value now to consider how the diagram shown in Fig. 2.5.1 may be used to obtain a computer solution. Since the user must implement the amplifier gains by means of resistors and capacitors, the first step leading to a computer circuit is to draw a more complete diagram which may be called the circuit diagram. This is shown in Fig. 2.5.2

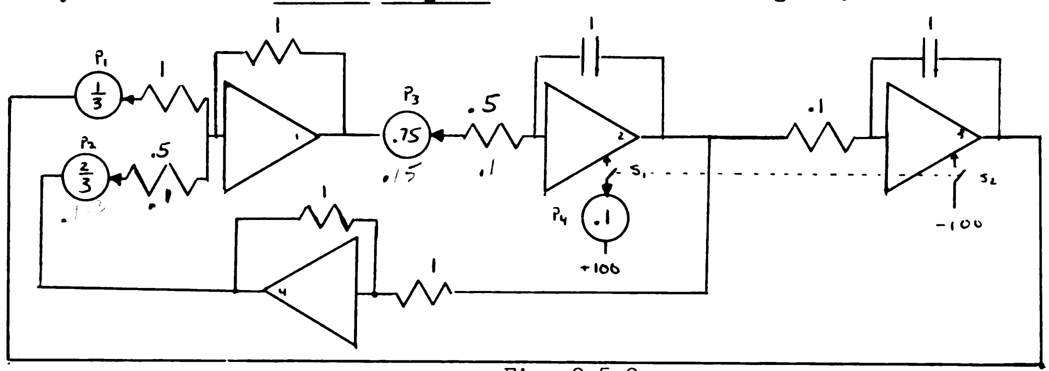


Fig. 2.5.2

Fig. 2.5.2 assumes that potentiometers and resistances of .1, .5, and 1 megs. and capacitances of 1 microfarad are available. All the amplifiers and potentiometers should be numbered with the same numbers used in the computer. In this diagram it is assumed that the amplifiers reverse the sign of the initial condition voltages.

Operating Procedure

After the circuit has been connected as shown in the figure, switches s_1 and s_2 are closed simultaneously so that the initial conditions may be applied. The voltage at the output of each amplifier should then be measured to be sure that all the connections and gains have been properly set. In this particular example, it is easy to verify that the amplifier outputs should be:

Amplifier 1	46.666	volts
" 2	-10.0	volts
" 3	100.0	volts
" 4	10.0	volts

The solution is then obtained by opening switches s_1 and s_2 simultaneously, permitting the analog blocks to operate on voltages appearing at their inputs. Since there are no constraints on the output voltages of these blocks, as soon as s_1 and s_2 are opened they will change in accordance with the operational equations. The output voltage of each block is now measured, and if any of the four amplifier outputs exceeds the maximum values specified by the manufacturer, the display variables are changed and the equation is re-scaled. For example, if the output of amplifier 1 exceeds the maximum value specified by the manufacturer, we may change the scale factor of the second derivative and display $\ddot{x}(t)/5$ rather than $\ddot{x}(t)/3$. This would naturally require a change in the gains of amplifiers 1 and 2.

Another difficulty that might occur is that, although none of the amplifiers is overloaded, one or more of them may not make use of their full range of operation. For example, if the first derivative swings from -5 to +5 volts, we might consider re-scaling the problem so that a larger swing is obtained.

If none of the amplifiers is overloaded and if all the voltages are of the proper order of magnitude, the solution is ready for recording. Fig. 2.5.3 shows the results of the solution given by the diagram of Fig. 2.5.2.

The recorder used in this case was a stripchart recorder. Notice that each trace was properly labeled with the following:

1) The name of the scaled variable. This permits the conversion between volts and the proper problem units.
2) The chart speed. This permits the proper interpretation of the time axis.
3) The time scale. This provides information in the case of a solution which may have been slowed down or speeded up as explained in paragraph 2.6.

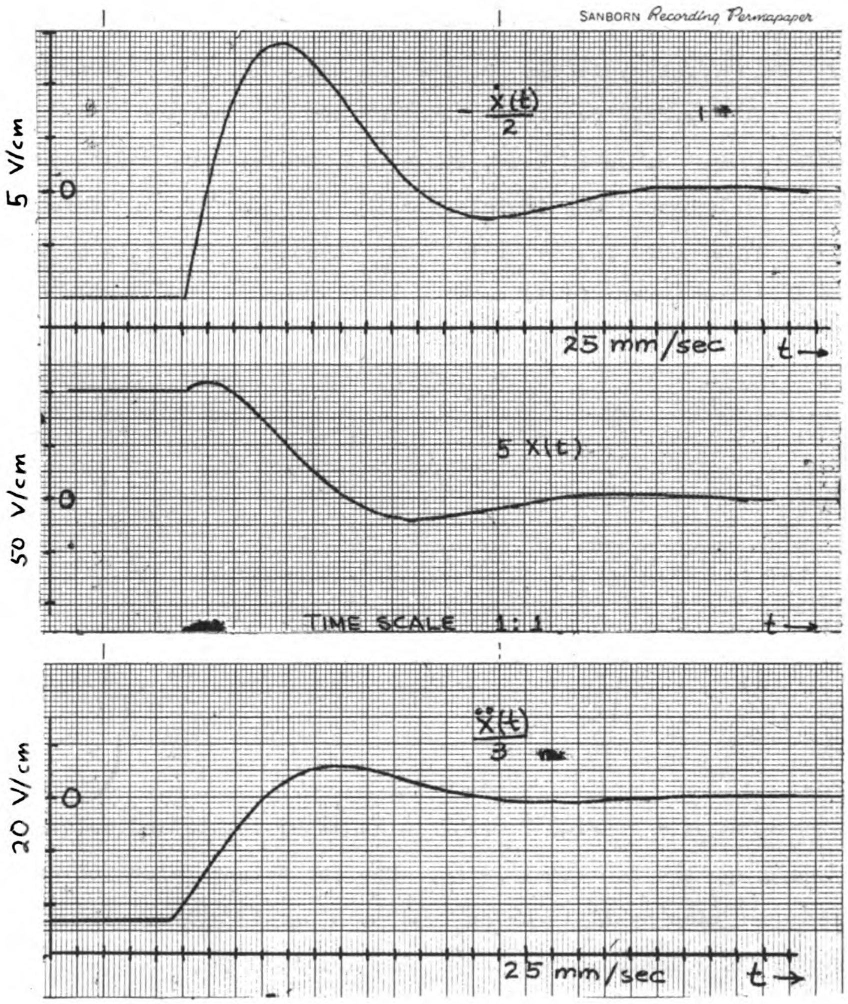


Fig. 2.5.3

2.6 Changing the Time Scale

The solutions obtained in an analog computer are time dependent, that is, the dependent variables and their derivatives are changing with respect to time. An important property of these variables is their frequency of oscillation, the common units of frequency being the cycles/second and the radians/second.

The frequency of oscillation is very important because each component of an electronic analog computer has a certain frequency range within which it will operate with the accuracy specified by the manufacturer. For example, an adder or an integrator may operate properly from zero frequency (constant voltages) to 500 cycles per second, but a strip-chart recorder might only be able to operate from 0 to 30 cycles per second.

Consider the computer solution of Fig. 2.6.1a. We notice that the frequency of oscillation was so high that it is difficult to separate one cycle of oscillation from another. In such a solution, we might want to be able to measure the values of the variable during the first cycle of operation. This first cycle was obtained in Fig. 2.6.1b by slowing down the solution so that instead of displaying the first cycle in 1 second of computer time, we display it in 10 seconds of computer time. Since it takes longer to display the same part of a solution, we say the computer has been slowed-down, in this case by a factor of 10. This is similar to slowing down the motor in a movie projector in order to see a picture in slow motion. It takes longer to see the same number of frames.

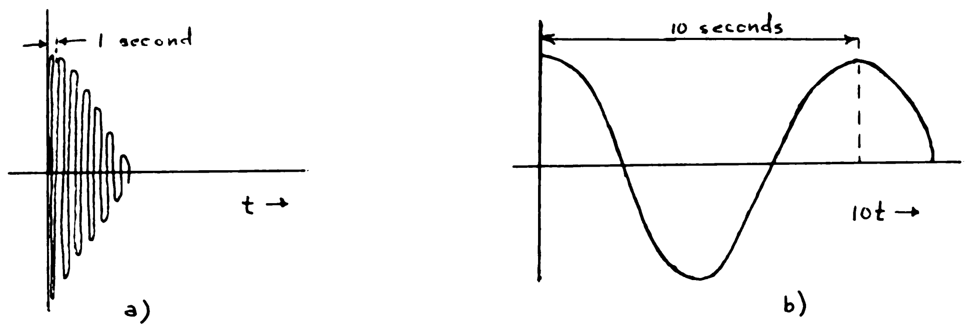


Fig. 2.6.1

Notice that the time axis of Fig. 2.6.1b is labelled 10t to indicate that the values of time read from the graph are ten times larger than the actual values of problem time. The user must divide computer time by 10 in order to obtain problem time. If computer time is called τ and problem time t, the relationship between them is given by

$$\tau = 10t$$

Thus, to obtain problem time t, the relation used is

$$t = \frac{\tau}{10} .$$

The solution shown in Fig. 2.6.2a, on the other hand, may be so slowly varying, that we might have to run the computer for five or ten minutes to get all of the values needed. This would be the case, for example, if we were studying some chemical process which has a slow rate of reaction. In this case we should speed-up the solution as shown in Fig. 2.6.2b.

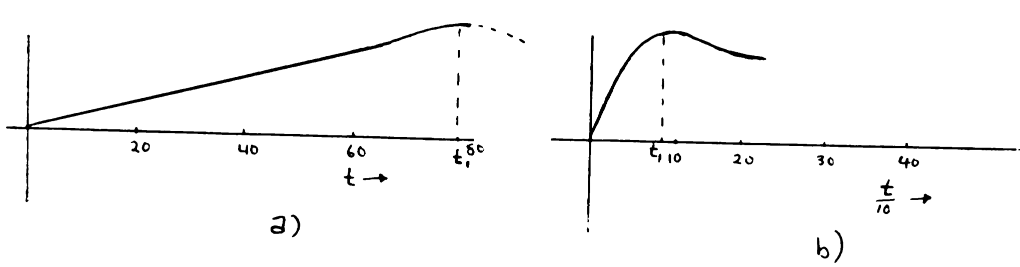


Fig. 2.6.2

In order to compare these figures, a reference time t_1 was marked in the time axis. In the unscaled solution shown in Fig. 2.6.2a it would take the computer 80 seconds to reach this reference time. In the scaled solution of Fig. 2.6.2b it would take the computer only 8 seconds to reach the same reference. The solution is then said to have been speeded-up by a factor of 10. In order to indicate this fact, the time axis of this solution is marked t/10 so that when the chart is read, we may realize that time must be multiplied by 10 in order to obtain actual problem time.

Since many physical problems are such that the frequencies involved exceed the acceptable ranges in the computer, the scale of the independent variable, time, must be changed in order to solve these problems.

The simplest method of changing the time scale is to proceed as follows:

1) The equation is magnitude-scaled by using the estimated maximum values of the dependent variables. This was explained in 2.5.

2) The frequencies of oscillation of the variables are estimated from knowledge of the physical system or they are observed after running the un-scaled solutions on the computer.

3) If the frequencies of oscillation are not within the operating range of the computer components, the time scale is changed as follows:

   a) If the frequencies are too high, the computer solution may be slowed-down by a factor K if the gains of all the integrators are divided by K.

   b) If the frequencies are too low, the computer solution may be speeded-up by a factor K if the gains of all the integrators are multiplied by K.

For example, if the frequencies of oscillation in a physical problem are of the order of 100 cycles per second and the available recorder is only accurate to 10 cps, the solution may be slowed-down by dividing the gain of every integrator by 10. This may be done by increasing either the capacitance or the resistance by a factor of 10.

If, on the other hand, the problem involves a very slow process which has an oscillating frequency of .01 cycles per second, the computer solution may be speeded-up by a factor of 100 by multiplying the gains of the integrators by 100. This may be done by decreasing either the capacitance or resistance of each integrator by a factor of 100.

In some analog computers the value of the integrating capacitors may be increased or decreased by a factor of 10 by merely throwing a switch. This simplifies time scaling considerably, since the user may speed-up or slow-down a solution with the least amount of algebraic manipulations. It should be remembered that a change in time scale by this method does not change the form of the variables displayed by the integrators. For example, if the output of an integrator is $(\ddot{x}(t)/k)$, it remains the same although the input-output relationship with respect to time of the integrator is different.

2.7 Estimation of Maximum Values

As was seen in the previous sections, scaling depends on the maximum values of the dependent variable and its derivatives and on the maximum value of their frequencies.

The most successful method of scaling results when the maximum values are estimated from our intimate knowledge of the sytem which we are trying to simulate in the computer. For example, an automotive engineer studying an automobile suspension system knows from previous experience the approximate maximum value of the displacement, velocity, and acceleration of an automobile tire. An electrical engineer familiar with an electrical network is able to make very good estimates of the maximum voltages and currents involved in the system.

Whenever the scientist is confronted with a physical system with which he is not familiar or which has not yet been built, the problem of scaling becomes one of wits and trial and error.

Although there is no general method for estimating the maximum values of the analog variables, a few methods which are successful in the case of simple linear equations have been worked out.* Since all methods are based on some rather restricting assumptions, they may or may not work when applied to a particular equation.

It is interesting to investigate the relationship between the scale factors and the amplifier gains. For example, the system

$$\left.\begin{aligned} a_2\ddot{x}(t) + a_1\dot{x}(t) + a_0x(t) &= 0, \\ x(0) &= x_0 \ , \\ \dot{x}(0) &= \dot{x}_0 \ , \end{aligned}\right\} \qquad (2.7.1)$$

may be scaled by choosing display variables $\left(\frac{x(t)}{s_0}\right)$, $\left(\frac{\dot{x}(t)}{s_1}\right)$, and $\left(\frac{\ddot{x}(t)}{s_2}\right)$ where s_0, s_1, s_2 are scale factors which are usually chosen from knowledge of the maximum values of the variables. This yields the system of (2.7.2).

$$\left.\begin{aligned} \left(\frac{\ddot{x}(t)}{s_2}\right) + \frac{s_1}{s_2}\frac{a_1}{a_2}\left(\frac{\dot{x}(t)}{s_1}\right) + \frac{s_0}{s_2}\frac{a_0}{a_2}\left(\frac{x(t)}{s_0}\right) &= 0 \\ \left(\frac{x(0)}{s_0}\right) &= \frac{x_0}{s_0} \ , \\ \left(\frac{\dot{x}(0)}{s_1}\right) &= \frac{\dot{x}_0}{s_1} \ . \end{aligned}\right\} \qquad (2.7.2)$$

The analog diagram for this equation is given in Fig. 2.7.1. If this is compared to the diagram of Fig. 2.3.3, which corresponds to the unscaled system of (2.7.1), we notice that each gain, except that of the sign changer, is a function of the scale factors s_0, s_1, and s_2. The initial conditions are also a function of these factors.

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*For example see Analog Computer Techniques, Clarence L. Johnson, McGraw-Hill Book Co., New York, 1956, and

Analog Computation, Albert S. Jackson, McGraw-Hill Book Co., New York, 1960.

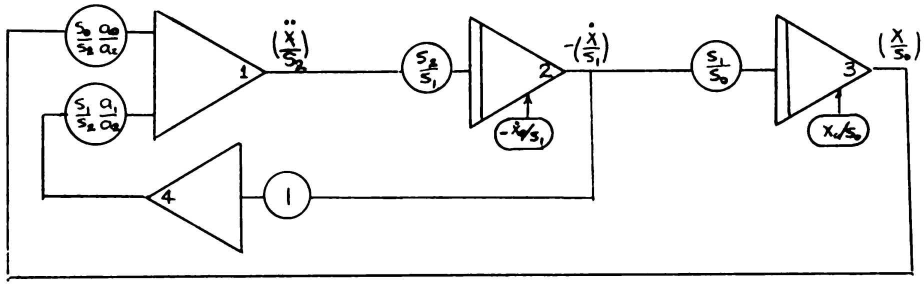


Fig. 2.7.1

This suggests a very convenient method for scaling a differential equation by trial and error. The method may be stated as follows:

1. Obtain the general diagram of the differential equation in terms of the scale factors.
2. Assume values for s_0, s_1, and s_2 from the estimated maximum values. Use these values in Fig. 2.7.1 to obtain a computer circuit for the first trial.
3. Three things may happen:
   a) No amplifier is overloaded. In this case measure the analog variables to see if they are of the proper magnitude for full accuracy. If they are, the trial and error procedure is finished. If one or more of the variables are too small for accuracy, change the scale factor and make another trial.
   b) One or more amplifiers are overloaded. In this case change the scale factor of the variables which overload an amplifier and make another trial.
   c) No amplifier is overloaded but the frequency of oscillation of the variables is either too high or too low. In this case we proceed as in a, except that the integrator gains are multiplied or divided by a constant in order to change the time scale.

<u>Example 2.7.1.</u>

In this example we will solve equation (2.5.1) by trial and error. Since this equation is a special case of (2.7.1), the general diagram of Fig. 2.7.1 applies.

In the first trial, we will assume that $s_o = s_1 = s_2 = 1$. When these values are written in Fig. 2.7.1, together with $a_o = 5$, $a_1 = 2$, and $a_2 = 1$, we obtain the diagram shown in Fig. 2.7.2.

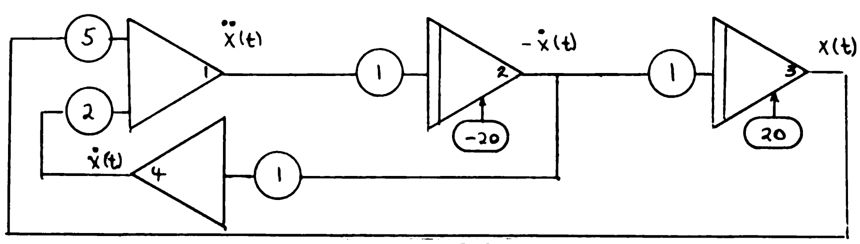


Fig. 2.7.2

If the circuit diagrammed in Fig. 2.7.2 is implemented on an analog computer, amplifier 1 will be overloaded. The reader should verify this with the computer at his disposal.

The fact that an amplifier is overloaded does not give us much information, since we are still not able to find the maximum value the amplifier is trying to reach. The trial and error procedure then requires that we choose a scale factor for $\ddot{x}(t)$. Suppose we choose $s_1 = 5$. Substituting this value into Fig. 2.7.1 we obtain Fig. 2.7.3

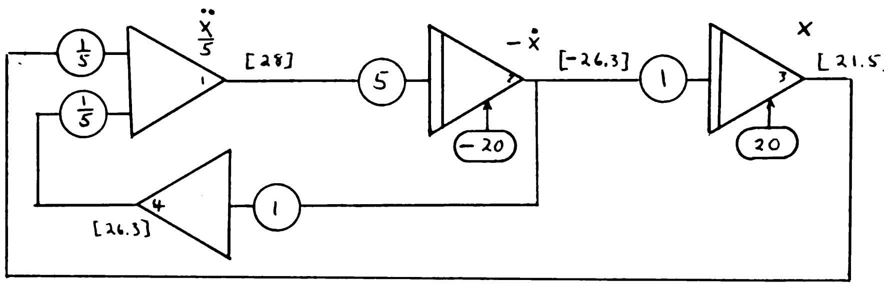


Fig. 2.7.3

The numbers in brackets represent the maximum value of each output as measured on the computer circuit. Since no amplifier is overloaded, this circuit may be used for the solution. Alternately, these maximum values may be used to obtain coefficients which allow the computer to use more of the range of amplifiers 1 and 2. Such a solution may be obtained if $s_o = \frac{1}{4}$, $s_1 = \frac{1}{4}$, and $s_2 = 2$.

2.8 Differential Equations with Forcing Functions

If we add a forcing function to (2.3.1), we obtain

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = f(t). \qquad (2.8.1)$$

Writing this equation in "highest derivative" and dot notation, we obtain

$$\ddot{y}(t) = -\left[\frac{a_1}{a_2}\dot{y}(t) + \frac{a_0}{a_2} y\right] + \frac{1}{a_2} f(t) \qquad (2.8.2)$$

This is almost identical to (2.3.7), except that a factor proportional to the forcing function appears on the right side of the equation. The "highest derivative method" then requires that an additional input equal to $-\frac{1}{a_2} f(t)$ be connected to the adder which produces $\ddot{y}(t)$. This input may be generated by a function generator as shown in Fig. 2.8.1.

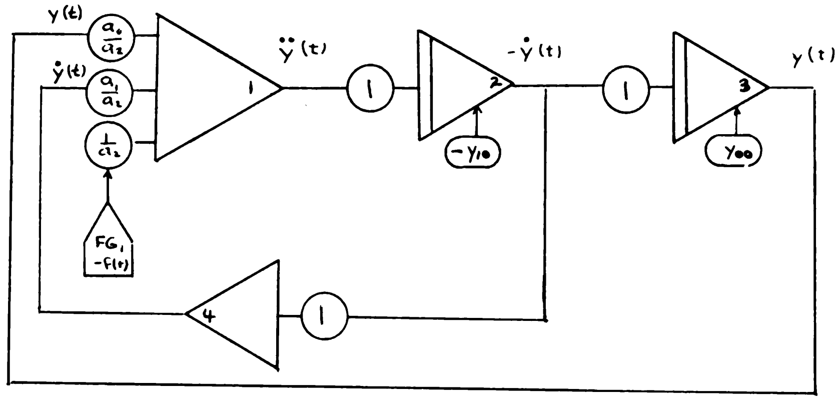


Fig. 2.8.1

Fig. 2.8.2 shows the diagram for the equation

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = f\left(\frac{dy}{dt}\right) \qquad (2.8.3)$$

in which the forcing function depends on the first derivative $\frac{dy}{dt}$. The idea may be extended to a forcing function which depends on one or more other variables.

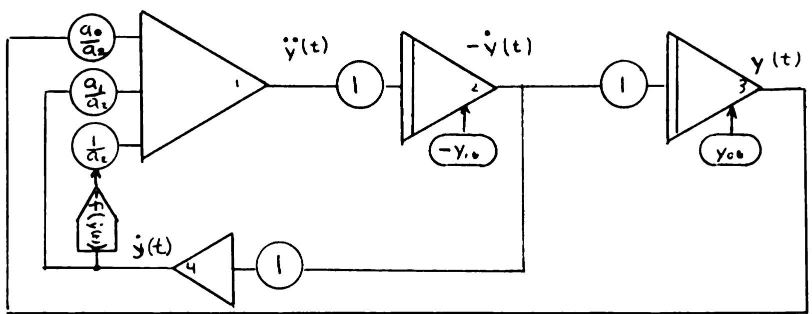


Fig. 2.8.2

It should be remembered that the forcing function f(t) must also be speeded-up or slowed-down whenever the time scale in the computer is changed. For example, if the computer time scale is slowed-down by a factor of 10, the function generated by the function generator must have a frequency which is one-tenth the given problem frequency.

2.9 Equations with Variable Coefficients

The equation

$$a_2 \ddot{y}(t) + a_1 \dot{y}(t) + f(t) \cdot y(t) = 0 \qquad (2.9.1)$$

has a coefficient f(t) which is a function of time. If the usual procedure is followed, the diagram shown in Fig. 2.9.1 results.

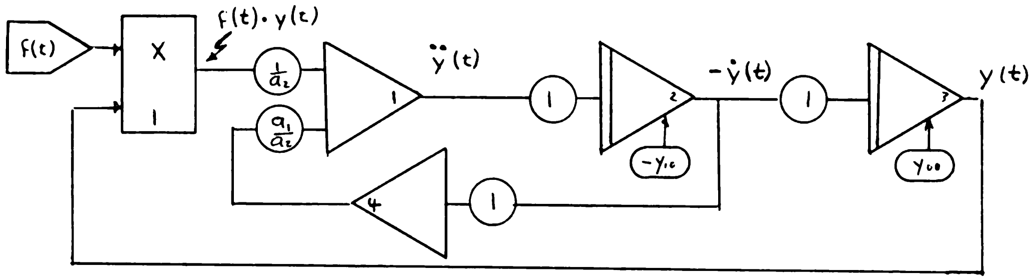


Fig. 2.9.1

Notice that since both f(t) and y(t) are time dependent, a function multiplier is required to obtain their product. If f(t) is an arbitrary function of time, a function generator is also needed to generate this function. If f(t) happens to be one of the available functions $\ddot{y}(t)$, $-\dot{y}(t)$, or y(t), the function generator may be eliminated. If f(t) is an integral power of t, such as t, t^2, t^3, etc., it may be generated by means of integrators as shown in Fig. 2.9.2.

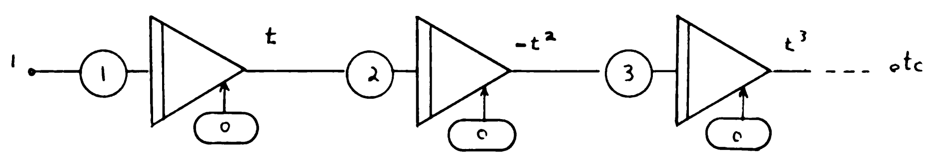


Fig. 2.9.2

2.10 Simultaneous Differential Equations

Simultaneous differential equations may be solved very conveniently by analog methods. The techniques used are the same as those used in the case of a single equation. For example, in the system

$$\left.\begin{aligned} a_{11}\ddot{y}(t) + b_{11}y(t) - b_{12}x(t) &= 0 \\ a_{22}\ddot{x}(t) - b_{21}y(t) + b_{22}x(t) &= 0 \end{aligned}\right\} \qquad (2.10.1)$$

$$\left.\begin{aligned} y(0) &= y_o, & \dot{y}(0) &= 0 \\ x(0) &= x_o, & \dot{x}(0) &= 0 \end{aligned}\right\} \qquad (2.10.2)$$

we notice that the top equation in (2.10.1) is a second order equation in y, except that the coupling term $-b_{12}x(t)$ also appears. Similarly, the lower equation is a second order equation in x except for the $-b_{21}y(t)$ term. The method used here is to find the analog circuit for each equation as if the other did not exist, and then couple these individual circuits by means of the coupling terms. For example, if we start with the equation in y, isolating the highest derivative, we obtain Fig. 2.10.1.

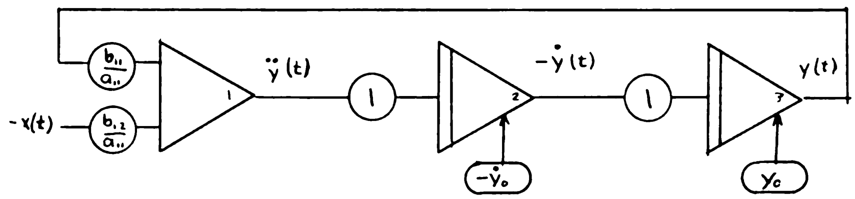


Fig. 2.10.1

If we proceed in the same way with the equation in x, the diagram shown in Fig. 2.10.2 results.

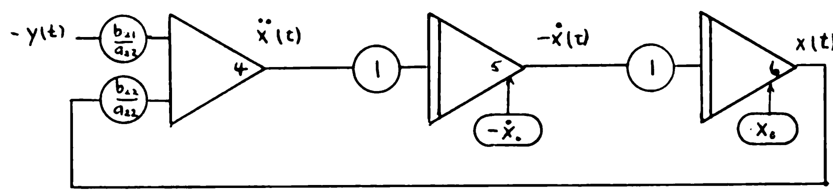


Fig. 2.10.2

The two diagrams are almost complete, except that -x(t) must be supplied to the y-system and -y(t) to the x-system. These are the coupling terms and voltages proportional to them are available in their corresponding diagrams. We notice, however, that they appear with the wrong sign, that is, the y-system provides a voltage analogous to y(t), but the x-system needs -y(t). The same difficulty appears in the case of the x(t) term. This may be resolved by assuming that $-\ddot{x}(t)$ is available at the output of amplifier 4 rather than $\ddot{x}(t)$. This changes the signs of all the analog voltages of Fig. 2.10.2 as seen in the complete diagram of Fig. 2.10.3.

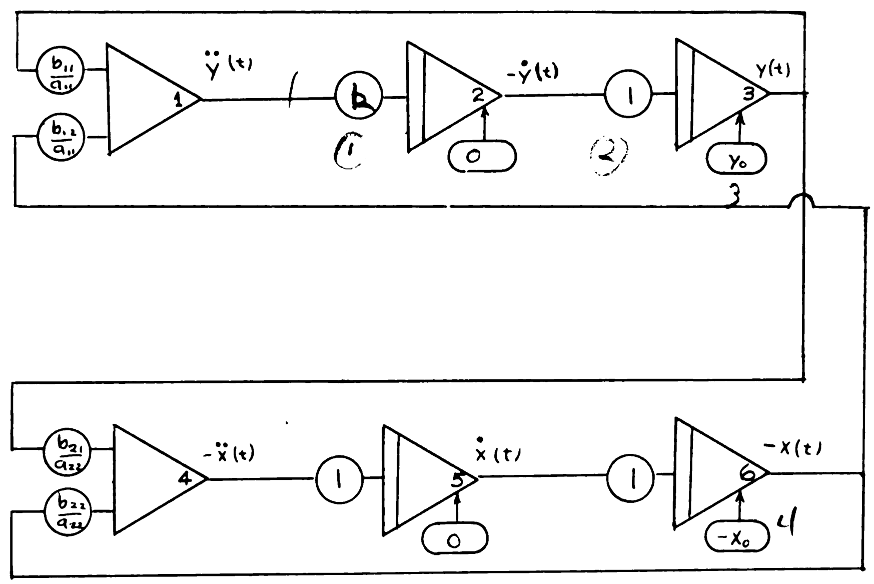


Fig. 2.10.3

It should be remembered that the diagram of Fig. 2.10.3 is the unscaled diagram of the system. When an actual problem is solved, scaling must be considered.

PROBLEMS

2.1 Draw the "highest derivative" logical diagram for the solution of the differential equation

$$a_0x(t) - a_1\dot{x}(t) + \dddot{x}(t) = a_3f(t)$$

$$\text{with } x(0) = x_{00}, \quad \dot{x}(0) = x_{01}, \quad \ddot{x}(0) = x_{02}$$

2.2 Draw the logical diagram for

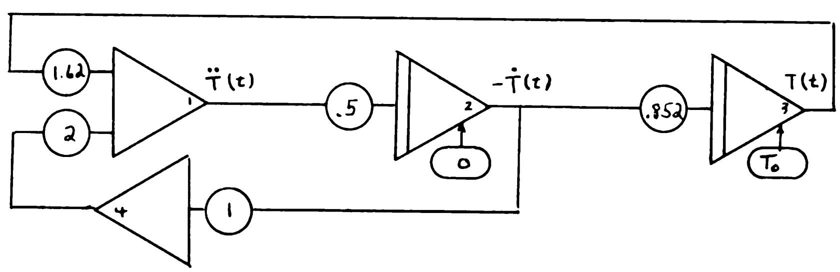
$$3\ddot{y}(t) - 2\dot{y}(t) + \dot{x}(t) = 0$$

$$\ddot{x}(t) - 3\dot{x}(t) - \dot{z}(t) = 0$$

$$2\ddot{z}(t) - \dot{z}(t) - \dot{y}(t) = 20$$

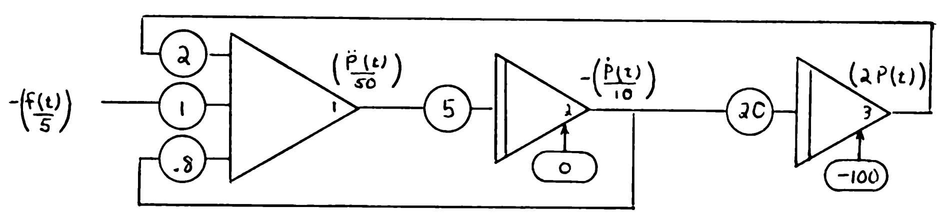
with all initial conditions equal to 0. How may we simplify the diagram if only the derivatives $\dot{x}(t)$, $\dot{y}(t)$, and $\dot{z}(t)$ are desired?

2.3 Draw the circuit diagram for the logic diagram given below. Assume that resistors of .1, 1, and 10 megaohms and capacitors of .1, 1, and 10 microfarads are available, and use a maximum of two potentiometers.

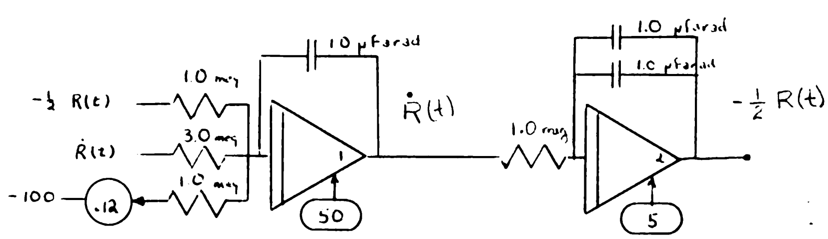


2.4 Write the differential equations and initial conditions for the diagrams below

a)



b)



2.5 The following questions apply to the diagram of Problem 2.4 (a.)

a. What is the output of each amplifier at $t = 0$ if $f(0) = -500$?

b. What gains must be used in amplifiers 2 and 3 to slow down the solution by a factor of 5?

c. What would happen if the function generator which produces $f(t)$ is momentarily disconnected from the input of amplifier 1 at $t = 0$?

2.6 Solve the equations

$$\ddot{x}(t) + 3x(t) - 2y(t) = 0$$
$$\ddot{y}(t) - 6x(t) + 7y(t) = 0$$

with initial conditions

$$x(0) = y(0) = 0$$
$$\dot{x}(0) = 10$$
$$\dot{y}(0) = 30$$

2.7 Solve the equations

$$\ddot{x}(t) + 4\dot{y}(t) + 9x(t) = 0$$
$$\ddot{y}(t) - 4\dot{x}(t) + 9y(t) = 0$$

with initial conditions

$$x(0) = \dot{y}(0) = 0$$
$$y(0) = \dot{x}(0) = -1$$

2.8 Solve the equations

$$3\ddot{y}(t) + 702y(t) - 4000x(t) = 0$$
$$\ddot{x}(t) - 7y(t) + 800x(t) = 0$$

and

$$y(0) = 50 \quad , \quad \dot{y}(0) = 0$$
$$x(0) = 2 \quad , \quad \dot{x}(0) = 0$$

Chapter 3. The Analog Computer

3.1 Introduction

Just as the size of a digital computer is often measured by its memory unit, number of input-output devices, etc., there are two important factors in estimating the size and computation power of an analog computer. These factors are:

a) the number of operational amplifiers

b) the number of non-linear elements such as multipliers and function generators.

Small analog computers might have from 5 to 30 operational amplifiers and few, if any, non-linear elements. Large analog computers have hundreds of amplifiers and dozens of non-linear devices. The number of resistors, capacitors, potentiometers increases with the number of amplifiers. In some special applications the output equipment such as display and recording devices become quite specialized and may comprise a large percentage of the cost of the computer. The cost of analog computers may be as low as $1000 and as high as hundreds of thousands of dollars depending on the size of the installation. A good educational analog computer with 5 to 10 amplifiers, 10 to 20 potentiometers, 2 or 3 mon-linear elements such as function generators or multipliers, and a number of resistance and capacitance elements may be bought for about $2000-$4000.

The schematic diagram of a small analog computer is given in Fig. 3.1.1. This computer does not exist, but it is very similar to most computers of its size. Most of the differences are in the manner in which the components are arranged.

Because of the many variationsin the equipment used in analog computers of different sizes and manufacturers, it is not within the scope of these notes to describe individual computer components in detail. It is important to the reader, however, to have a general understanding of what these components are. A brief description of the computer of Fig. 3.1.1 will be used to give the reader this understanding. After reading this description, the reader should consult the operating manual of the particular computer which is available to him and should locate and study its components parts.

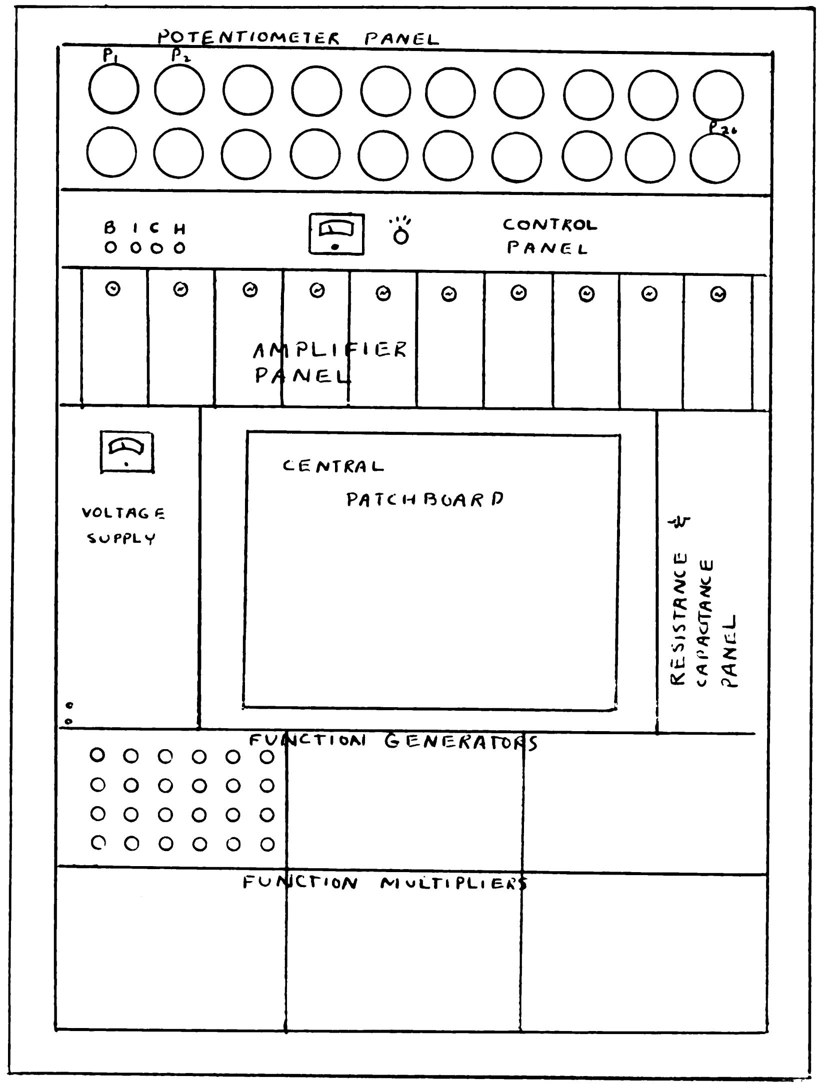


Fig. 3.1.1

3.2 The POTENTIOMETER PANEL

This panel houses the ten-turn potentiometers and their corresponding dials. Each potentiometer is marked by a label such as P1, P2, etc., for identification. The three leads of each potentiometer may be brought out to plug-in jacks below it or to jacks at the central patch-board. Some of the pots may have one of their terminals internally grounded, that is, connected to the common computer ground. Then only the input and output (slider) terminals are brought out to plug-in jacks. In some computers it may be possible to apply the computer's reference voltage, usually $\pm$ 100 volts, to the input terminal of any potentiometer so that the attenuator constant may be set by reading the output voltage on a voltmeter as seen in Fig. 3.2.1.

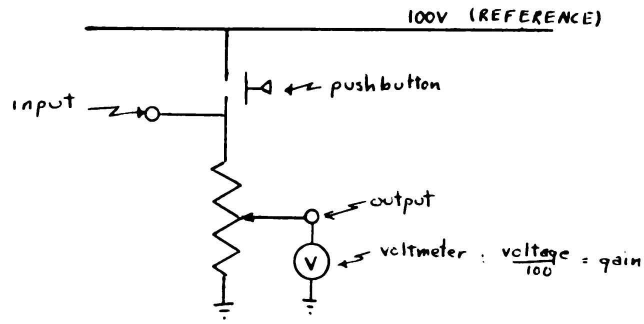


Fig. 3.2.1

3.3 The AMPLIFIER PANEL

This section houses the operational amplifiers which are needed for implementing the analog blocks. Each amplifier may be a self-contained unit which may be un-plugged from the computer for replacement or repair, or two or more amplifiers may be part of a multiple unit.

The individual amplifiers may be of the multi-purpose type that may be used for any of the blocks discussed in previous paragraphs or they may be pre-wired as sign-changers, adders, integrators, etc.

The front panel of a general purpose plug-in amplifier might be as shown in Fig. 3.3.1. The neon light is used to warn the user that the output voltage has exceeded the maximum or minimum

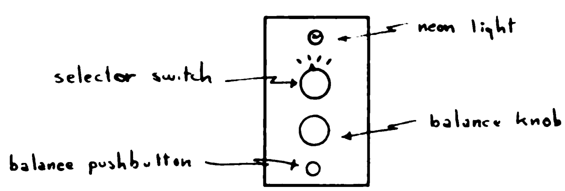


Fig. 3.3.1

allowable voltage. Whenever the neon light is ON, the amplifier does not operate properly and it is said to be overloaded. In this case the problem must be re-scaled.

The selector switch may be used to select the mode of operation of the amplifier, that is, its use as an adder, an integrator, etc., or it may be absent in certain designs.

The balance adjustment knob is used in adjusting the electronic circuit so that a zero input voltage will produce a zero output voltage. This operation is called "balancing the amplifier."

Some amplifiers are of the type shown in Fig. 3.3.2a where both polarities of the output are provided. This eliminates the need for sign-changers in the analog circuits at the expense of more equipment inside each individual block, since the amplifier in this case is equivalent to a regular amplifier and a sign-changer as shown in Fig. 3.3.2b .

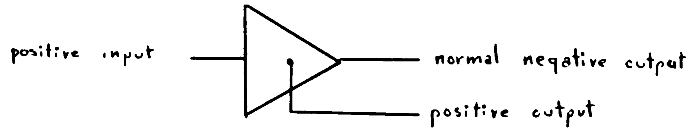


Fig. 3.3.2a)

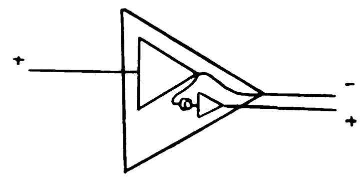


Fig. 3.3.2b)

As will be seen in Chapter 4, amplifiers of the double-output type would simplify the programming of analog computers if a matrix approach is used. The extra expense involved, however, has discouraged most manufacturers from offering computers using this concept.

3.4 The CONTROL PANEL

The most important function of the control panel is to provide a centralized location from which all the computer components are at the command of the operator.

The four push buttoms marked BAL, INITIAL, COMP, and HOLD are connected to a number of internal relays and switches which perform the following functions: When the BALANCE buttom is pressed the inputs of all the amplifiers are connected to the common ground (0 volts) by special networks. The operator may then adjust the amplifier output voltage to zero by turning the balance adjustment knob shown in Fig. 3.3.1. When the INITIAL button is pressed, the initial condition voltages applied to the initial condition terminals of the integrators are sent to the integrating capacitors. Pushing the COMPUTE button releases the initial condition voltages and permits the analog blocks to operate according to their operational equations. When the HOLD button is activated, all the integrating capacitors are disconnected from the integrators so that the capacitors will not be discharged as long as the computer is in the HOLD position. This is equivalent to "stopping time," since all of the voltages in the analog circuit will remain static if the integration process is halted by disconnecting the integrating capacitors.

The HOLD condition permits the operator to make changes in the circuit, such as changing potentiometer settings. This is equivalent to changing the problem parameters after a solution has been partially run.

In our model computer the control panel also houses a dc voltmeter and a meter scale switch. One side of the meter is connected to the ground reference. By means of a toggle switch the other side may be connected to either an external jack or to a meter bus. A meter bus is nothing but a good conductor brought to the balance push button of each amplifier as shown in Fig. 3.4.1.

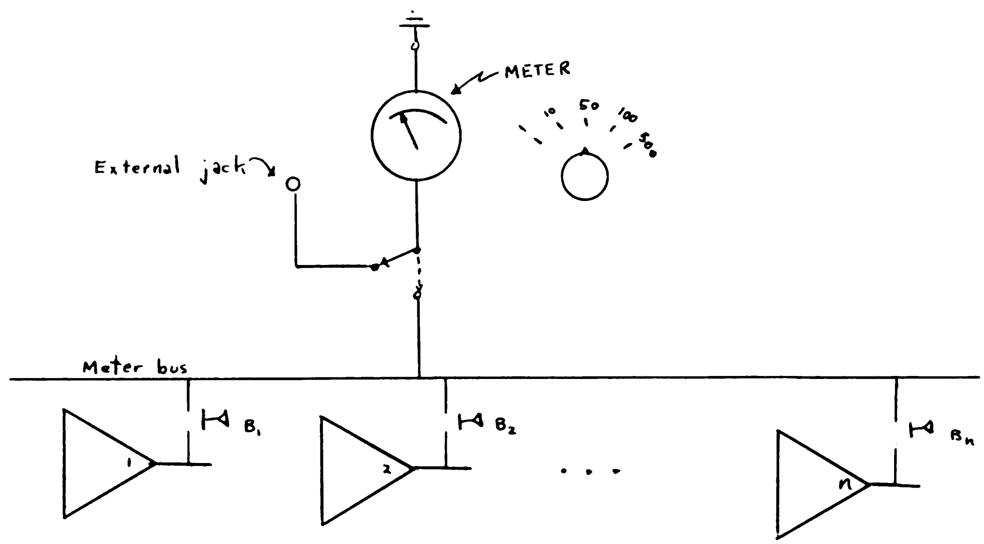


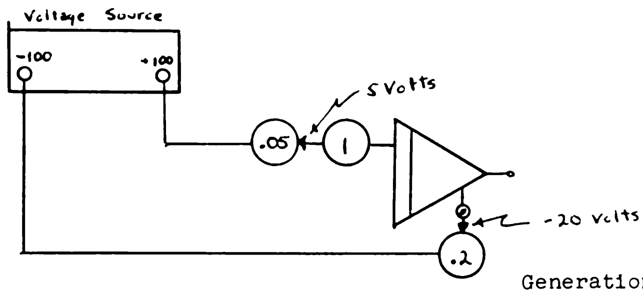
Fig. 3.4.1

The meter may then be used as 1) a regular meter through its external jack, 2) a balance meter when the computer is in its BALANCE mode, 3) a checking device when the amplifier is in INITIAL, or 4) a measuring device when the amplifier is in COMPUTE. In every case, except the first, only the output of the amplifier whose push button is being activated will be measured.

In some computers the meter is of the type called "digital voltmeter" which displays the voltages in digital form rather than with a pointer and a scale.

3.5 The VOLTAGE SUPPLY PANEL

The voltage suply panel houses electronically regulated power supplies which provide very stable voltages for use as constant voltages and initial condition voltages. In some cases the voltage supply panel provides only the reference voltages, usually $\pm$ 100 volts, and these have to be scaled down with potentiometers as shown in Fig. 3.5.1. In other cases the voltage supply panel has built-in attenuators so that voltages of varying magnitudes and polarities may be obtained directly.



Generation of f(t)=20-5t

Fig. 3.5.1

3.6 RESISTANCE AND CAPACITANCE PANEL

This panel contains the resistors and capacitors which are used in the amplifiers. If the amplifiers are of the multi-purpose type which already contain pre-wired resistors and capacitors, the panel is used to complement the R-C components already available. The terminals of all components are either brought out to plug-in jacks or to the central patch board. In the latter case, the patch board is clearly labeled with the value of each resistance in megohms and each capacitance in microfarads.

3.7 FUNCTION GENERATOR PANEL

As the name implies, this is where one or more function generators are located. Depending on the method used for the generation of functions, the panel might contain a number of knobs for setting the parameters of the function to be generated.

3.8 FUNCTION MULTIPLIER PANEL

This panel contains the function multipliers and all the necessary meters and adjustments for the calibration and operation of these devices.

3.9 The PATCH BOARD

The patch board is that part of the computer where all the computer elements are connected to each other by means of patch cords. The input and output terminals of all the amplifiers, resistors, capacitors, potentiometers, etc., are all wired internally to plug-in jacks at the patchboard. The face of the board is clearly labeled to simplify the operation of the machine. In some computers the patch board is removable, so that it may be wired while the computer is being used by someone else. If several patch boards are available, several people may be getting ready to use the computer at the same time. It is a simple matter to insert a pre-wired patch board into the computer and to run the solution in a matter of minutes.

Problems which are run frequently may be kept in separate patch boards ready to be inserted whenever they are needed. This is similar to having a library of digital computer programs on punched cards or magnetic tape.

A removable plug board permits more efficient use of the computer because the machine is used only when the operator is ready to run his problem and it is not tied up while the components are being interconnected or when wiring errors are being traced.

3.10 OUTPUT EQUIPMENT

The most common pieces of output equipment are the strip-chart recorder, the X-Y plotter, and the oscilloscope.

The strip-chart recorder is nothing but a dc voltmeter whose pointer is equipped to leave a trace on a moving strip of accurately calibrated paper. In some cases the pointer is similar to an ink pen which leaves an ink trace as it moves across the chart. In other cases the pointer has a heating element which burns a fine line on specially treated paper. In both cases the recorder will look similar to Fig. 3.10.1.

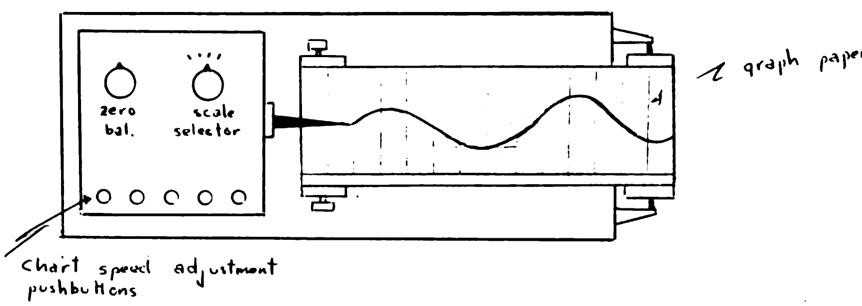


Fig. 3.10.1

The recorder in Fig. 3.10.1 is a one-channel recorder which permits the recording of only one variable at a time. In most computers multi-channel recorders which may record several variables at once are available.

The X-Y plotter is a device which has two degrees of freedom. The inking pen moves in the X direction (x-axis) as a function of the voltage applied between the ground terminal and the X input terminal. The pen moves in the Y direction as a function of the voltage applied between the ground terminal and the Y input terminal. In this fashion, the final location of the pen on an X-Y coordinate system will depend on the magnitude of the X and Y voltages. A drawing of an X-Y recorder is shown in Fig. 3.10.2.

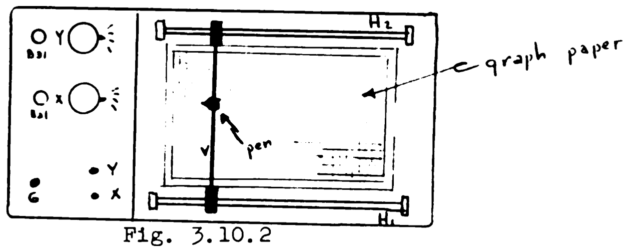


Fig. 3.10.2

As may be seen from the figure, the vertical bar V which carries the pen may move right and left by riding on the horizontal guides H_1 and H_2. At the same time, the pen may move up and down by riding on the vertical bar. These motions are controlled by the voltages applied to the X and Y inputs respectively. Each input has a corresponding scale selector switch and a balance adjustment which positions the pen anywhere in the paper when both input voltages are zero.

If a voltage is to be plotted versus real time, the arrangement of Fig. 3.10.3a may be used. In this case an integrator with -1 volt applied to its input generates the time function t which may be applied to the X-axis. The voltage to be plotted is then applied to the Y-axis.

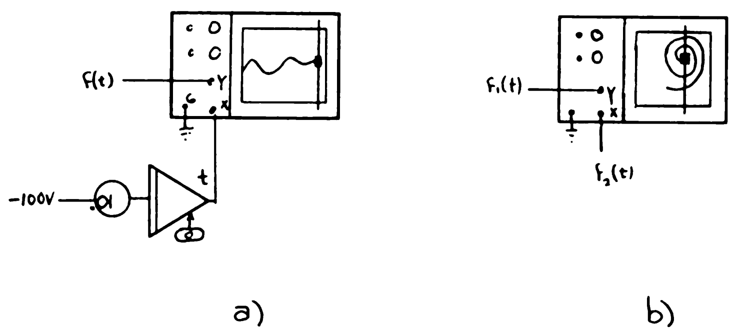


Fig. 3.10.3

In other cases we may wish to plot one function of time $f_1(t)$ versus another $f_2(t)$. This may be done as shown in Fig. 3.10.3b . This second arrangement is not available in the strip chart recorder, since the x-axis is always moved at constant speed by the strip chart drive motor.

An oscilloscope is a device which operates in a manner similar to the X-Y recorder, except that it has no mechanical parts and is all electronic. The plotting pen is now an electron beam which moves in the X-Y plane as a function of the voltages applied to the X and Y inputs. Instead of having a piece of paper for recording the answer, the oscilloscope produces a spot of light on a fluorescent tube similar to the tube in a TV set. As the spot moves across the tube, a lighted trace is produced which is visible for a fraction of a second. In this way the performance of the X-Y plotter is duplicated at electronic speeds. The lighted trace may be photographed by a special oscilloscope camera if the result is to be preserved.

Since the oscilloscope is all electronic, it operates at higher frequencies than the electromechanical X-Y plotter. For this reason, the oscilloscope is used in high speed analog computers of the repetitive type. These computers have special circuits which permit them to repeat the same solution several times each second. If the same trace is repeated on an oscilloscope several times a second, the observer may see the complete solution without having to wait until an inking pen traces across the paper. In this manner, parameters may be changed while the computer is displaying the solution on the oscilloscope. The effect of parameter changes on the solution is therefore immediately apparent on this type of computer.

Chapter 4 MATRIX PROGRAMMING OF ANALOG COMPUTERS

4.1 Introduction

In previous chapters we were concerned with the use of analog blocks for the solution of linear and differential equations. The analog blocks were put together by using the highest derivative method and other methods which rely on our intuition. Although we were aware of the fact that several computer circuits could be found for the solution of each problem, it was difficult to tell if the circuit which we designed could be improved.

In this chapter we will show a straight forward method based on the theory of matrices which will permit us to optimize analog circuits, to eliminate unnecessary operational amplifiers, and to find the relationship between equivalent circuits. In addition, the methods in this chapter will organize our work in such a way that our design of a good circuit will not depend as much on intuition. Furthermore, we will find that familiarity with the matrix methods in this chapter will widen our understanding of the solution of equations in general, and will bring to light the relationship between mathematics and the analog networks.

Although a basic background in the theory of matrices will make some of the methods that follow more meaningful, the reader should not feel handicapped in reading this chapter if he has not been introduced to matrix algebra. The chapter will be self-contained in the sense that all of the necessary rules are introduced within the section in which they are needed. Actually, we could eliminate the word "matrix" from this chapter and consider that we are dealing with tables or arrays of numbers which are to be manipulated according to a number of rules and procedures which yield the desired results.

In the next few sections a few concepts of matrix algebra are reviewed very briefly. Only those concepts which are relevant to the treatment of matrix programming of analog computers are covered. A minimum number of definitions and only a few new terms are introduced. The reader should consult texts on matrix algebra for a formal treatment of the subject.

4.2 The Algebra of Matrices

In this chapter we will encounter arrays of numbers or functions which may be arranged into m rows and n columns in the form of an m x n table which may be represented mathematicly by the symbol

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \qquad (4.2.1)$$

and may be called a matrix. A matrix of m rows and n columns is called a m x n matrix, and the numbers or functions which form the matrix are called the elements of the matrix. The element in the ith row and jth column of the matrix is identified symbolically as a_{ij}. The elements a_{11}, a_{22}, ... , a_{mm} are said to belong to the diagonal of the matrix.

Most of the matrices in this chapter are of two types:

a) square matrices in which m=n, and

b) column matrices in which n=1.

Examples of square matrices are given in Fig. 4.2.1

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} (a+b) & 1 \\ 5 & (b+c) \end{bmatrix}$$

(a) (b) (c)

Fig. 4.2.1

The matrix shown in Fig. 4.2.1a is a 3 x 3 matrix in which every element is equal to zero. A matrix in which every element $a_{ij}=0$ is called a null or zero matrix, thus this matrix is a 3 x 3 zero matrix. Matrix b has each diagonal element equal to unity and each non-diagonal element equal to zero, and is a special case of the unit matrix, in this case a 4 x 4 unit matrix. Matrix c is a 2 x 2 matrix in which some of the elements are expressions of the algebraic variables a, b, and c.

Examples of column matrices are given in Figure 4.2.2

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \qquad \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

a) b) c)

Fig. 4.2.2

The column matrix shown in Fig. 4.2.2a is a column matrix of real constants, matrix b is a column matrix of variables x_1, x_2, ... , x_6, and matrix of c is a column matrix of functions of time. Column matrices are sometimes called vectors.

The manipulation of matrices is simplified if they are given names or symbols. Matrices are usually named with letters of the alphabet. In these notes, boldface capital letters will be reserved for naming matrices. For example, the unit matrix will be called by the name **I**, and the zero matrix by the name **O**. Other matrices will be named with the letters **A**, **B**, **C**, **D**, etc.

The set of rules, theorems, postulates, etc., for manipulating matrices is called the algebra of matrices or matrix algebra. It is beyond the scope of these notes to give even a brief treatment of this useful and interesting subject. It will suffice to state some simple definitions and to review the rules of addition, subtraction, and multiplication of matrices.

4.3 Equality of Matrices

Two matrices **A** and **B** are equal if and only if:

a) they have the same number of rows and columns (they are of the same order)

b) every element a_{ij} of **A** is equal to the corresponding element b_{ij} of **B**

When two matrices **A** and **B** are equal, we symbolize their equality by the matrix equation

$$\mathbf{A} = \mathbf{B} \tag{4.3.1}$$

Example 4.3.1

The matrices

$$\mathbf{A} = \begin{bmatrix} a+b & 1 \\ d & c \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$$

are equal if and only if $a + b = 3$, $d = 5$, and $c = -1$.

4.4 Addition of Matrices

The sum of two m x n matrices

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ b_{m1} & b_{m2} & & b_{mn} \end{bmatrix}$$

may be defined by the matrix

$$\mathbf{C} = \begin{bmatrix} (a_{11}+b_{11}) & (a_{12}+b_{12}) & \cdots & (a_{1n}+b_{1n}) \\ (a_{21}+b_{21}) & (a_{22}+b_{22}) & \cdots & (a_{2n}+b_{2n}) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (a_{m1}+b_{m1}) & (a_{m2}+b_{m2}) & \cdots & (a_{mn}+b_{mn}) \end{bmatrix} \tag{4.4.1}$$

in which

$$c_{ij} = a_{ij} + b_{ij} \tag{4.4.2}$$

We may write symbolically

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{4.4.3}$$

According to this definition, it is obvious that $\mathbf{A}$ and $\mathbf{B}$ must have the same number of rows and columns, i.e., both matrices must be of the same order. Similarly, the operation

$$\mathbf{C} = \mathbf{A} - \mathbf{B} \tag{4.4.4}$$

is defined by the equation

$$c_{ij} = a_{ij} - b_{ij} \tag{4.4.5}$$

Addition of matrices which are equal leads to the concept of multiplicity. For example, the addition

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 2a_{11} & 2a_{12} & 2a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{bmatrix} \tag{4.4.6}$$

suggests the notation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 2 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{4.4.7}$$

in which the 2 may be thought of as a coefficient of multiplicity, or, in other words, the coefficient which indicates how many equal matrices are to be added. If we adopt this notation, the following equation may be used to define it:

$$K \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & & a_{nm} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & & a_{nm} \end{bmatrix} K = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1m} \\ ka_{21} & ka_{22} & \cdots & ka_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ ka_{n1} & ka_{n2} & & ka_{nm} \end{bmatrix}$$

or in other words, the same constant may be "factored out" of <u>every</u> <u>element</u> of a matrix.

4.5 Multiplication of Matrices

The product of an m x n matrix $\mathbf{A}$ and an n x p matrix $\mathbf{B}$ is the m x p matrix $\mathbf{C}$ defined by the equation

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \,. \qquad (4.5.1)$$

If the operations indicated by (4.5.1) are done by hand, it is convenient to organize the work as indicated in Fig. 4.5.1. In this figure an element of $\mathbf{C}$ is found by adding the products of the row and column which intersect at the element. For example, the c_{12} term is the point

$$\begin{bmatrix} b_{11} & b_{12} \cdots & b_{1p} \\ b_{21} & b_{22} \cdots & b_{2p} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} \cdots & b_{np} \end{bmatrix}$$

Fig. 4.5.1

$$\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix} \qquad \begin{bmatrix} c_{11} & c_{12} \cdots & c_{1p} \\ c_{21} & c_{22} \cdots & c_{2p} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ c_{m1} & c_{m2} \cdots & c_{mp} \end{bmatrix}$$

of intersection between row 1 of $\mathbf{A}$ and column 2 of $\mathbf{B}$ so its value is obtained as shown below

$$\begin{array}{l} b_{12} \\ b_{22} \\ \cdot \\ \cdot \\ \cdot \\ \downarrow b_{n2} \end{array}$$

Fig. 4.5.2

$$\underrightarrow{a_{11} \quad a_{12} \cdots \quad a_{1n}} \qquad c_{12} = a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2}$$

Again, we may see from Fig. 4.5.1 that the process of generating each element of the product is only possible if the matrix used as the multiplier has as many columns as the matrix used as the multiplicand has rows. Therefore, the product of matrices is more restricted than the product of ordinary numbers. However, it should be obvious that the product of two square matrices of the same order is always possible.

Example 4.5

Find **A·B** if $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -2 & 5 \\ 3 & 5 & 0 \end{bmatrix}$

$$\begin{array}{cc} & \begin{bmatrix} 1 & -2 & 5 \\ 3 & 5 & 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} & \begin{bmatrix} (2+15) & (-4+25) & (10+0) \\ (1+9) & (-2+15) & (5+0) \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 17 & 21 & 10 \\ 10 & 13 & 5 \end{bmatrix}$$

The most common type of matrix multiplication encountered in these notes is the product of a square matrix and a column matrix or vector. In this case matrix multiplication is very simple since it becomes the sum of the products of the elements of each row of the matrix times the elements of the vector, as shown below for a 3 x 3 array.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} \end{bmatrix}$$

Fig. 4.5.3

The multiplication is simplified if the diagram shown in Fig. 4.5.4 is used. Where the brackets enclosing the vector indicate that the vector is really a <u>column</u> vector which has been written in row form.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \left\{ b_{11} \quad b_{21} \quad b_{31} \right\} = \mathbf{C}$$

Fig. 4.5.4

If Figure 4.5.4 is then written as shown in Fig. 4.5.5, it is easy to see that matrix multiplication becomes the sum of the product of each element of the matrix with its corresponding element of the vector

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{Bmatrix} b_{11} & b_{21} & b_{31} \end{Bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} \end{bmatrix}$$

Fig. 4.5.5

4.6 Matrix Equations and the Ideograph

Consider the equation

$$a_o y + a_1 \frac{dy}{dt} + a_2 \frac{d^2y}{dt^2} = 0. \qquad (4.6.1)$$

We may define the operators

$$s = \frac{d}{dt}, \quad s^2 = \frac{d^2}{dt^2}, \quad \ldots, \quad s^n = \frac{d^n}{dt^n} \qquad (4.6.2)$$

so that

$$\frac{dy}{dt} = sy, \quad \frac{d^2y}{dt^2} = s^2y, \qquad (4.6.3)$$

and the equation may be rewritten in operator notation

$$a_o y + a_1 s^1 y + a_2 s^2 y = 0. \qquad (4.6.4)$$

If we make the coefficient of the highest derivative equal to 1, we obtain

$$\frac{a_o}{a_2} y + \frac{a_1}{a_2} sy + s^2y = 0. \qquad (4.6.5)$$

We will now show that this equation may be written in the matrix notation of (4.6.6), and that this notation will permit us to find an analog circuit which will yield the solution of the equation.

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ \frac{a_o}{a_2} & \frac{a_1}{a_2} & 1 \end{bmatrix} \cdot \begin{bmatrix} y \\ sy \\ s^2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (4.6.6)$$

In this equation the matrix

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ \frac{a_o}{a_2} & \frac{a_1}{a_2} & 1 \end{bmatrix}$$

is called the matrix of coefficients or <u>coefficient matrix</u>, the column matrix

$$\begin{bmatrix} y \\ sy \\ s^2y \end{bmatrix}$$

is called the solution matrix or <u>solution vector</u>, and the matrix to the right of the equal sign

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is called the matrix of <u>constant terms</u>.

This equation yields, according to our definition of matrix multiplication,

$$\begin{bmatrix} sy - sy + 0 \\ 0 + s^2y - s^2y \\ \frac{a_o}{a_2} y + \frac{a_1}{a_2} sy + s^2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (4.6.7)$$

The column matrices at each side of the equal sign in (4.6.7) are equal if and only if

$$\begin{aligned} sy - sy + 0 &= 0 \\ 0 + s^2y - s^2y &= 0 \\ \frac{a_o}{a_2} y + \frac{a_1}{a_2} sy + s^2y &= 0 \end{aligned}$$

These equations do not contain any contradictions and, in fact, generate the given differential equation. We will find that several matrix equations similar to (4.6.6) may be found which have the property of generating the given differential equation. For example, the reader should verify that (4.6.9), (4.6.10) and (4.6.11) are three such systems.

$$\begin{bmatrix} s & 1 & 0 \\ 0 & s & 1 \\ \frac{a_0}{a_2} & \frac{-a_1}{a_2} & 1 \end{bmatrix} \cdot \begin{bmatrix} y \\ -sy \\ s^2y \end{bmatrix} = 0 \qquad (4.6.9)$$

$$\begin{bmatrix} 0 & 1 & s \\ 1 & s & 0 \\ -a_0 & -a_1 & a_2 \end{bmatrix} \cdot \begin{bmatrix} -s^2y \\ sy \\ -y \end{bmatrix} = 0 \qquad (4.6.10)$$

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ a_0 & a_1 & a_2 \end{bmatrix} \cdot \begin{bmatrix} -sy^2 \\ -sy \\ -y \end{bmatrix} = 0 \qquad (4.6.11)$$

The reason for going through all the trouble of writing the differential equation in matrix form is that it has been shown (see references 1, 2. 3, 4) that it is possible to draw the analog circuit corresponding to the equation in such a way that there is a perfect correspondence between the matrix coefficients and the amplifier connections. This is done by drawing the analog circuit shown in Fig. 4.6.1. This figure, which is called an ideograph, may be drawn directly from the matrix equation.

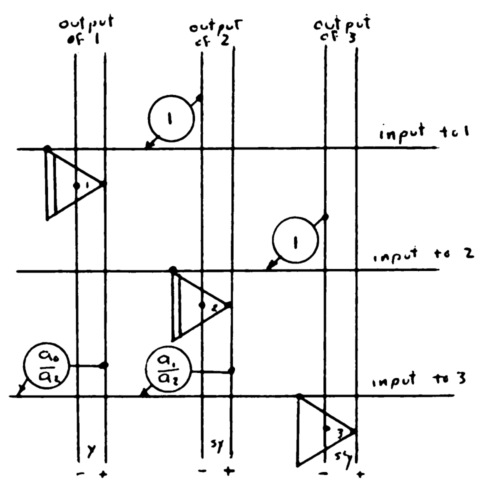
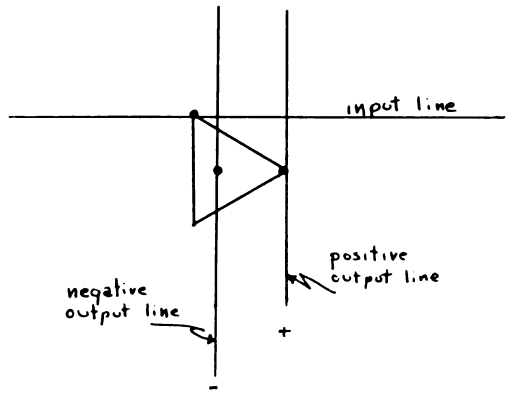


Fig. 4.6.1

In this figure there are three triangles representing amplifiers. These amplifiers are of the double-output type described in section 3.4. A detailed view of one of these amplifiers is given in Fig. 4.6.2.



Note:
There is a change in sign between the input line and the POSITIVE output line

Fig. 4.6.2

The main diagonal of the coefficient matrix is used to determine whether an amplifier is an integrator or an adder. If the diagonal element contains the s operator, the amplifier is an integrator, otherwise, it is an adder, a constant multiplier, or a sign-changer. The non-diagonal entries of the matrix correspond to the gains of the amplifiers. Since there are four non-zero off-diagonal elements, four circles with appropriate gains are shown.

The solution vector is translated into the label of the outputs at the bottom of the ideograph. Since double-output amplifiers are used, both y and -y, sy and -sy, and s^2y and $-s^2y$ are available for recording. Alternately, a single output amplifier may be used for amplifier 1, since only the normal positive output of this amplifier was needed.

Since most computer installations do not have double-output amplifiers, these may be simulated by a regular amplifier and a sign changer as shown in Fig. 4.6.3.

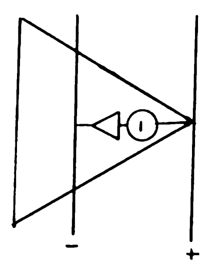


Fig. 4.6.3

As will be seen later, more efficient ideographs may be developed for single-output amplifiers, so that the reader should not be too concerned about this detail now.

The ideograph of Fig. 4.6.1 may be called the gain ideograph to differentiate it from those developed in reference 1[1]. It may be used very conveniently for making connections since it indicates very clearly the relationships between amplifier inputs, outputs, and gains. For example, if we consider the output of each amplifier in the order of amplifier

1 The gain ideograph was developed by the author as a natural extension of those developed by Honnell and Horn. It has the advantage of being less circuit oriented since it stresses the logical components of the analog circuit rather than its circuit components.

number, we find that the output of amplifier 1 is connected to one of the inputs of amplifier 3 with a gain of a_0/a_2. The negative output of amplifier 2 is connected to the input of amplifier 1, with a gain of 1. The positive output is connected to the input of amplifier 3 with a gain of a_1/a_2. The outputs of amplifier 3 are connected in the same manner. When all the outputs have been considered, all the inputs will be properly connected. Of course, the same result will be produced if the inputs, rather than the outputs, are connected in sequence.

Some readers might prefer to draw the familiar block diagram corresponding to the ideograph shown in Fig. 4.6.1. This has been done in Fig. 4.6.4, It is somewhat inconvenient because we usually draw the block diagrams so that the derivatives decrease in power from left to right. The next section will show an ideograph which removes this inconvenience.

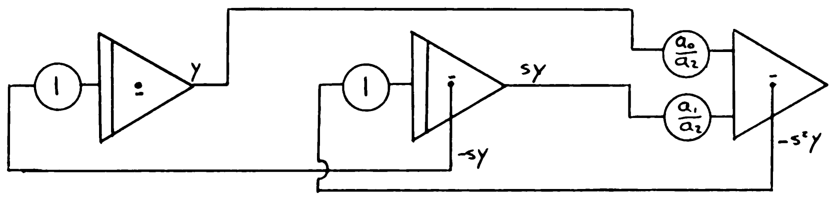


Fig. 4.6.4

We should also notice that an array which approximates the ideograph even more closely may be obtained by placing the solution vector below the matrix of coefficients as shown in Fig. 4.6.5.

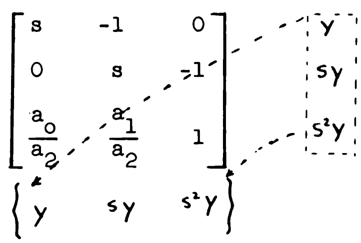


Fig. 4.6.5

In this case the solution vector is enclosed in { } to warn the reader that such a transformation has been applied.

4.7 The Reflected Ideograph

A more natural computer diagram, in which the highest derivative appears at the left side of the diagram, may be obtained by using the matrix equation

$$\begin{bmatrix} 0 & -1 & s \\ -1 & s & 0 \\ 1 & \frac{a_1}{a_2} & \frac{a_0}{a_2} \end{bmatrix} \cdot \begin{bmatrix} s^2y \\ sy \\ y \end{bmatrix} = 0 \qquad (4.7.1)$$

which may be obtained from (4.6.6) by finding the mirrow image of the coefficient and columns matrices as shown below.

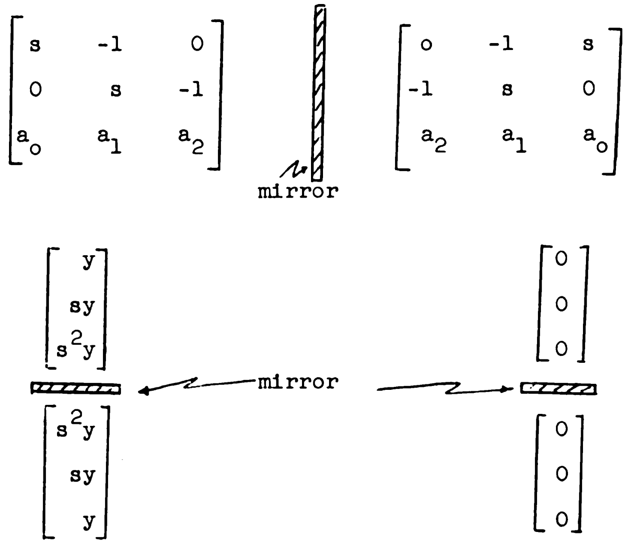


Fig. 4.7.1

When equation (4.7.1) is written in the form of (4.7.2) from which we may obtain Fig. 4.7.2, it is easy to see the correlation between the matrix equation and the ideograph of Fig. 4.7.3.

$$\begin{bmatrix} 0 & -1 & s \\ -1 & s & 0 \\ 1 & \frac{a_1}{a_2} & \frac{a_0}{a_2} \end{bmatrix} \cdot \left\{ s^2y \quad sy \quad y \right\} = \mathbf{0} \qquad (4.7.2)$$

$$\begin{bmatrix} 0 & -1 & s \\ -1 & s & 0 \\ 1 & \frac{a_1}{a_2} & \frac{a_0}{a_2} \end{bmatrix}$$

$$\cdot$$

$$\left\{ s^2y \quad sy \quad y \right\}$$

$$=$$

$$\mathbf{0}$$

Fig. 4.7.2

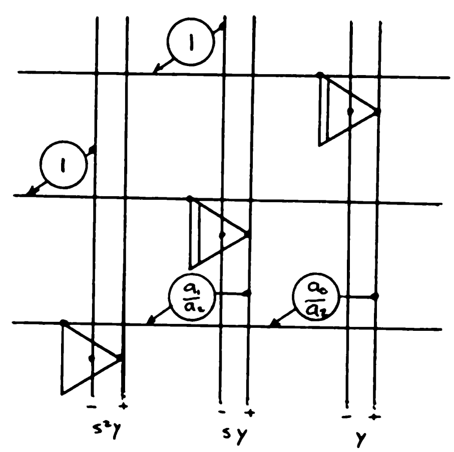


Fig. 4.7.3

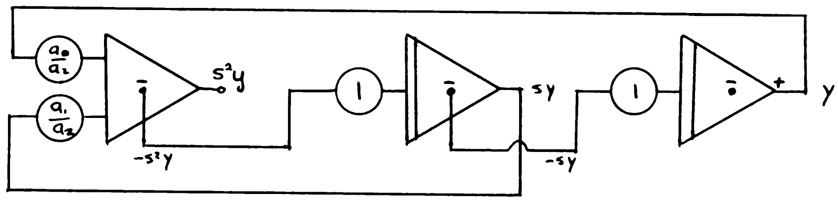


Fig. 4.7.3 cont'd

A question which might come to the reader's mind is: What happens if the systems of (4.6.9), (4.6.10), or (4.6.11) are used as models for the ideograph? The answer is that each equivalent matrix equation would yield an ideograph with perhaps a different appearance and, in general, a different number of amplifiers.

The techniques developed in these notes will assume the use of a matrix of a particular form as the starting point of the procedure of drawing the ideograph. It will then be shown that one system may be converted to another by suitable matrix manipulations.

4.8 Standard Form of Ordinary Differential Equations with Constant Coefficients

In order to convert the differential equation

$$a_o x + a_1 sx + a_2 s^2 x + \ldots + a_{n-1} s^{n-1} + s^n = 0 \qquad (4.8.1)$$

into a matrix equation, we may use the standard form of (4.8.2).

$$\begin{bmatrix} s & -1 & 0 \ldots 0 & 0 \\ 0 & s & -1 \ldots 0 & 0 \\ 0 & 0 & s \ldots 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 \ldots s & -1 \\ a_o & a_1 & a_2 \ldots a_{n-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \\ s^2x \\ \cdot \\ \cdot \\ \cdot \\ s^{n-1}x \\ s^n x \end{bmatrix} = \mathbf{0} \qquad (4.8.2)$$

In this system, all elements along the diagonal except the one in the lower right hand corner are equal to s. To the right of each s there is a -1, The last row consists of the coefficients of the differential equations. The column matrix of variables consists of all the derivatives of x in ascending order from top to bottom.

Equation (4.8.2) leads to an ideograph in which the derivatives increase in order from left to right. An ideograph in which the derivatives decrease from left to right may be found by reflecting (4.8.2) to obtain(4.8.3).

$$\begin{bmatrix} 0 & 0 \; . \; . \; . \; 0 & -1 & s \\ 0 & 0 \; . \; . \; . -1 & s & 0 \\ 0 & 0 \; . \; . \; . \; s & 0 & 0 \\ . & . \quad\quad . & . & . \\ . & . \quad\quad . & . & . \\ . & . \quad\quad . & . & . \\ -1 & s \; . \; . \; . \; 0 & 0 & 0 \\ 1 & a_{n-1} . \; . \; a_2 & a_1 & a_o \end{bmatrix} \cdot \begin{bmatrix} s^n x \\ s^{n-1} x \\ s^{n-2} x \\ . \\ . \\ . \\ sx \\ x \end{bmatrix} = \mathbf{0} \qquad (4.8.3)$$

In this system the operators appear in the other diagonal of the matrix, except the lower left hand position which contains a 1. To the left of each operator there is a -1, and the last row contains the coefficients of the differential equation. The solution vector consists of all the derivatives of x in descending order, from top to bottom.

In both (4.8.2) and (4.8.3) the differential equation is in standard operator notation, that is, the coefficient of the highest derivative has been adjusted to unity.

Most of the methods in this chapter are based on using (4.8.2) or (4.8.3) as the starting point and on transforming these equations into other equivalent systems by means of elementary matrix transformations. The advantage of these methods is that it is easier to manipulate matrices than to work with block diagrams, especially if this manipulation is being done by a digital computer program.

4.9 Elementary Matrix Transformations and Minimization of Sign-Changers.

Going back to matrix equations (4.6.6) and (4.7.1) we notice that a sign changer is needed for each negative entry in the coefficient matrix. This suggests the possibility of minimizing the number of sign changers by minimizing the number of negative matrix elements. Since, as was shown in section 4.6, there are many different matrix equations for each differential equation, it should be possible to find an equivalent system with fewer negative entries. One such system is given in (4.9.1) which has been written with (4.6.6) for comparison.

We can see that (4.9.1) may be obtained from (4.6.6) by changing the sign of all the elements in row 2 of the matrix equation and of all the elements in column 2 of the coefficient matrix. As a result of a double sign change, the diagonal element is not affected.

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ \frac{a_o}{a_2} & \frac{a_1}{a_2} & 1 \end{bmatrix} \cdot \begin{bmatrix} y \\ sy \\ s^2 y \end{bmatrix} = \mathbf{0} \qquad\qquad \begin{bmatrix} s & 1 & 0 \\ 0 & s & 1 \\ \frac{a_o}{a_2} & -\frac{a_1}{a_2} & 1 \end{bmatrix} \cdot \begin{bmatrix} y \\ -sy \\ s^2 y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(4.6.6) (4.9.1)

An operation such as this which does not affect the ability of the matrix equation to generate the given differential equation is called an "elementary matrix transformation."

Elementary matrix transformations (EMT) are important because they permit us to transform one matrix equation to another. Since each equation represents a different analog system, we have at our disposal a method by which we may choose a system, from all those possible, which is superior in some respect to all others. For example, if we wish to minimize the number of double-output amplifiers, we should change the sign of those rows and columns which contribute the largest number of negative entries in the coefficient matrix. Therefore, we may state the first elementary matrix transformation (EMT) as follows:

EMT 1. - "The signs of all the elements in any row of the matrix equation may be changed if at the same time the signs of all elements of the corresponding column of the coefficient matrix are changed."

Example 4.9.1

The systems below are equivalent (EMT 1 on row 3)

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} z \\ sz \\ s^2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} z \\ sz \\ -s^2z \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix}$$

Example 4.9.2

Minimize the number of sign-changers in the equation

$$\begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ 0 & 0 & s & -1 \\ 3 & 5 & -2 & 1 \end{bmatrix} \begin{bmatrix} y \\ sy \\ s^2y \\ s^3y \end{bmatrix} = \mathbf{0}$$

Solution: Applying EMT 1 to row-column 2 we obtain

$$\begin{bmatrix} s & 1 & 0 & 0 \\ 0 & s & 1 & 0 \\ 0 & 0 & s & -1 \\ 3 & -5 & -2 & 1 \end{bmatrix} \begin{bmatrix} y \\ -sy \\ s^2y \\ s^3y \end{bmatrix} = \mathbf{0}$$

Applying EMT 1 to row-column 4 we obtain

$$\begin{bmatrix} s & 1 & 0 & 0 \\ 0 & s & 1 & 0 \\ 0 & 0 & s & 1 \\ -3 & 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} y \\ -sy \\ s^2y \\ -s^3y \end{bmatrix} = 0$$

A word of warning is now in order. Elementary matrix transformations are easier to apply to a matrix in which the s operators appear along the main diagonal than to a reflected matrix. For example, if the matrix of Example 4.9.2 is reflected, we obtain

$$\begin{bmatrix} 0 & 0 & -1 & s \\ 0 & -1 & s & 0 \\ -1 & s & 0 & 0 \\ 1 & -2 & 5 & 3 \end{bmatrix} \begin{bmatrix} s^3y \\ s^2y \\ sy \\ y \end{bmatrix} = 0 \qquad (4.9.3)$$

Notice that columns should be counted from right to left and rows from top to bottom in the coefficient matrix. Rows in the solution vector should be counted from the bottom to the top. Changing the sign of row 3 may lead to error since the variable corresponding to row 3 does not line-up with row 3 of the coefficient matrix.

The best way to apply EMT 1 to a reflected matrix is to place the solution vector below the coefficient matrix and to change the sign of the variable corresponding to the column which was changed in sign.

In the matrix above we whould proceed as follows:

Changing signs of row-column 3 gives.

$$\begin{bmatrix} 0 & 0 & -1 & s \\ 0 & 1 & s & 0 \\ 1 & s & 0 & 0 \\ 1 & 2 & 5 & 3 \end{bmatrix}$$

$$\begin{Bmatrix} s^2y & -s^2y & sy & y \end{Bmatrix}$$

Changing signs in row-column 1,

$$\begin{bmatrix} 0 & 0 & 1 & s \\ 0 & 1 & s & 0 \\ 1 & s & 0 & 0 \\ 1 & 2 & 5 & -3 \end{bmatrix}$$

$$\begin{Bmatrix} s^3y & -s^2y & sy & -y \end{Bmatrix}$$

If this system is reflected, we obtain

$$\begin{bmatrix} s & 1 & 0 & 0 \\ 0 & s & 1 & 0 \\ 0 & 0 & s & 1 \\ -3 & 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} -y \\ sy \\ -s^2y \\ s^3y \end{bmatrix} = 0 \qquad (4.9.4)$$

which differs from the final result of Example 4.9.2 in that the signs of the elements of the solution vector are changed. This leads to EMT 2.

EMT 2

"Changing the sign of every element in the solution vector is an elementary matrix transformation."

Other elementary matrix transformations which are useful in finding equivalent analog networks will be introduced as they are needed in succeeding sections.

In order to show the use of (4.8.2), the following example is given.

Example 4.9.3 Solve the equation

$$7\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 5y = 0. \tag{4.9.5}$$

The matrix equation is then constructed:

$$\left(-\frac{5}{7} + \frac{2}{7}s - \frac{3}{7}s^2 + s^3\right)\cdot y = 0.$$

The matrix equation is then constructed:

$$\begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ 0 & 0 & s & -1 \\ -\frac{5}{7} & \frac{2}{7} & -\frac{3}{7} & 1 \end{bmatrix} \cdot \begin{bmatrix} y \\ sy \\ s^2y \\ s^3y \end{bmatrix} = \mathbf{0} \tag{4.9.6}$$

The use of EMT 1 on row-column 3 gives

$$\begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & 1 & 0 \\ 0 & 0 & s & 1 \\ -\frac{5}{7} & \frac{2}{7} & \frac{3}{7} & 1 \end{bmatrix} \cdot \begin{bmatrix} y \\ sy \\ -s^2y \\ s^3y \end{bmatrix} = \mathbf{0} \tag{4.9.7}$$

which should be verified by the reader as an equivalent system by performing the matrix multiplication.

Using EMT 1 again on row-column 1 we obtain

$$\begin{bmatrix} s & 1 & 0 & 0 \\ 0 & s & 1 & 0 \\ 0 & 0 & s & 1 \\ \frac{5}{7} & \frac{2}{7} & \frac{3}{7} & 1 \end{bmatrix} \begin{bmatrix} -y \\ sy \\ -s^2y \\ s^3y \end{bmatrix} = \mathbf{0} \tag{4.9.8}$$

The corresponding ideograph is given in Fig. 4.9.1

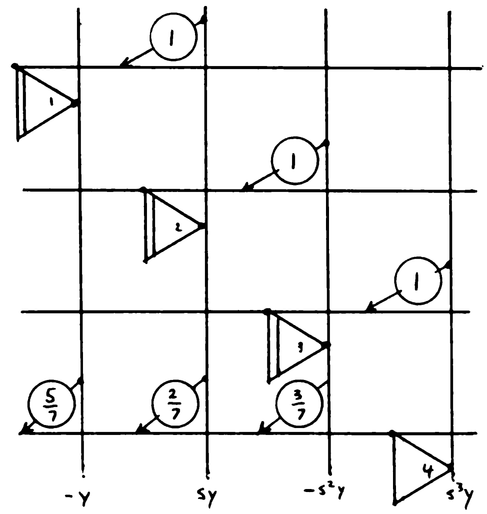


Fig. 4.9.1

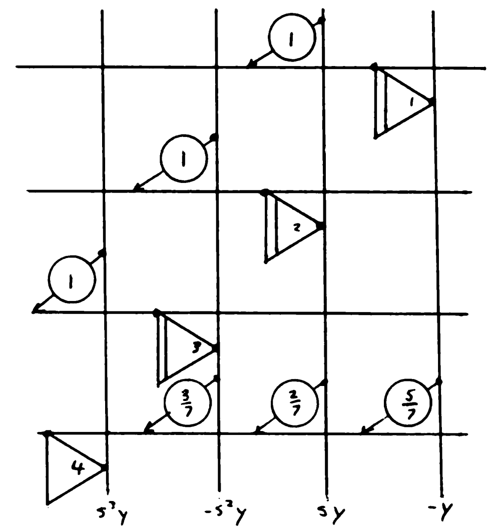


Fig. 4.9.2

The reflected gain ideograph of Fig. 4.9.2 may be obtained by using the mirrow image of (4.9.8)

$$\begin{bmatrix} 0 & 0 & 1 & s \\ 0 & 1 & s & 0 \\ 1 & s & 0 & 0 \\ 1 & \frac{3}{7} & \frac{2}{7} & \frac{5}{7} \end{bmatrix} \begin{bmatrix} s^3y \\ -s^2y \\ sy \\ -y \end{bmatrix} = 0 \qquad (4.9.9)$$

or the mirror image of the ideograph of Fig. 4.9.1

4.10 Strategies for the Reduction of Negative Entries

Whenever the number of variables is large and the coefficient matrix has many negative entries, it is not simple to decide what to do in order to minimize the number of negative entries unless some strategy is followed. We will show how we may do this by considering a larger matrix equation.

$$\begin{bmatrix} s & -1 & 0 & 0 & 0 & 0 \\ 0 & s & -1 & 0 & 0 & 0 \\ 0 & 0 & s & -1 & 0 & 0 \\ 0 & 0 & 0 & s & -1 & 0 \\ 0 & 0 & 0 & 0 & s & -1 \\ 2 & -3 & -4 & 6 & -5 & 7 \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \\ s^2x \\ s^3x \\ s^4x \\ s^5x \end{bmatrix} = 0 \qquad (4.10.1)$$

The first step is to prepare a "sign map" in which only the sign of the non-zero, non-diagonal entries, are shown. Also, the row numbers are carried on the right side of the map.

0	-	0	0	0	0	1
0	0	-	0	0	0	2
0	0	0	-	0	0	3
0	0	0	0	-	0	4
0	0	0	0	0	-	5
+	-	-	+	-	0	6
0	-3	-3	-1	-3	-2	

Fig. 4.10.1

The second step is to add the number of positive signs and subtract the number of negative signs of each row-column pair and to write these differences at the bottom of each column. For example, row-column 1 has a difference of 0, since there is one negative entry and one positive entry. On the other hand, row-column 4 has one positive and two negative for a difference of -1.

The third step is to change the sign of the row-column pair which contains the largest negative difference. Sometimes, as is the case in this example, there is more than one possibility. In this case any of the choices will produce the same final result.

Changing the sign of the second row and column we obtain

0	+	0	0	0	0	1
0	0	+	0	0	0	-2
0	0	0	-	0	0	3
0	0	0	0	-	0	4
0	0	0	0	0	-	5
+	+	-	+	-	0	6
2	3	-1	-1	-3	0	

Fig. 4.10.2

Notice that the sign of the row number was changed in order to remind us that the sign of the corresponding variable should be changed in the transformed matrix equation.

The new sign map has a maximum negative difference in column 5, so we proceed to change the sign of the fifth row and column. If we proceed this way, the two following consecutive maps are produced.

0	+	0	0	0	0	1
0	0	+	0	0	0	-2
0	0	0	-	0	0	3
0	0	0	0	+	0	4
0	0	0	0	0	+	-5
+	+	-	+	+	0	6
2	3	-1	1	3	4	

0	+	0	0	0	0	1
0	0	-	0	0	0	-2
0	0	0	+	0	0	-3
0	0	0	0	+	0	4
0	0	0	0	0	+	-5
+	+	+	+	+	0	6
2	1	1	3	3	6	

Fig. 4.10.3

As soon as all the differences are positive it is impossible to reduce the number of negative entries any further. We then use the sign map in order to obtain the matrix equation

$$\begin{bmatrix} s & 1 & 0 & 0 & 0 & 0 \\ 0 & s & -1 & 0 & 0 & 0 \\ 0 & 0 & s & 1 & 0 & 0 \\ 0 & 0 & 0 & s & 1 & 0 \\ 0 & 0 & 0 & 0 & s & 1 \\ 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix} \cdot \begin{bmatrix} x \\ -sx \\ -s^2x \\ s^3x \\ -s^4x \\ s^5x \end{bmatrix} = \mathbf{0} \qquad (4.10.2)$$

which requires a single sign-changer.

The strategy which we have used in the preceeding procedure is essentially an algorithm which may be programmed on a digital computer. The digital computer would accept the sign map of Fig. 4.10.1 and change the sign of columns and rows until the column differences are all positive.

4.11 Incorporation of Sign-Changers into the Ideograph

Suppose that a matrix equation has been transformed by a number of elementary transformations and we find that one or more entries are still negative. For example, the matrix equation

$$\begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ 0 & 0 & s & -1 \\ a_o & -a_1 & -a_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \\ s^2x \\ s^3x \end{bmatrix} = \mathbf{0} \qquad (4.11.1)$$

may be transformed into

$$\begin{bmatrix} s & 1 & 0 & 0 \\ 0 & s & -1 & 0 \\ 0 & 0 & s & 1 \\ a_o & a_1 & a_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ -sx \\ -s^2x \\ s^3x \end{bmatrix} = \mathbf{0} \qquad (4.11.2)$$

by suitable transformations. Any attempts to eliminate the remaining negative entry will bring no improvements. This means that at least one amplifier will need a sign-changer associated with it, namely the amplifier which yields $-s^2x$. This is shown in the gain ideograph of Fig. 4.11.1.

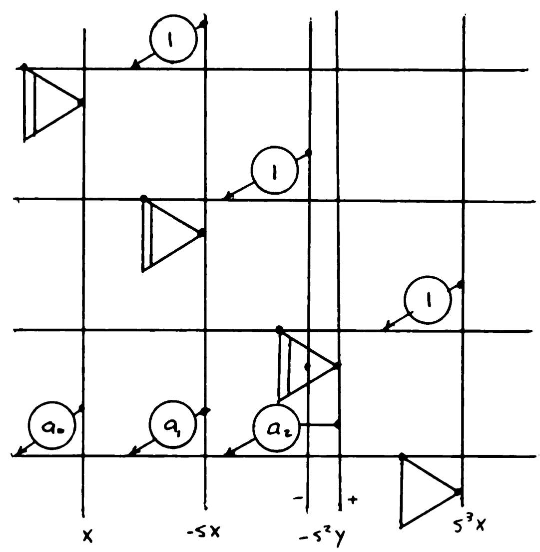


Fig. 4.11.1

A very simple procedure may be used for obtaining an ideograph in which all the amplifiers are of the single-output type and all the sign changers are incorporated in the ideograph. The procedure will be presented by applying it to equation (4.11.2)

1) By the use of elementary matrix transformations or by using the sign map, reduce the number of negative entries to a minimum (this has already been done in this example).

2) Augment the coefficient matrix by adding a new row-column pair. (in this example, row and column 5). The elements of this new pair are:

a) 1 in the diagonal position (5,5)

b) 1 in the <u>column</u> of the negative constant (5,3)

c) the negative constant, with its sign changed, in the <u>row</u> of the negative constant (2,5)

(If there is more than one negative constant in the same <u>column</u>, repeat step c)

d) zeros elsewhere.

$$\begin{bmatrix} s & 1 & 0 & 0 & 0 \\ 0 & s & -1 & 0 & 1 \\ 0 & 0 & s & 1 & 0 \\ a_0 & a_1 & a_2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

3) Place a zero in the position occupied by the negative constant(s).

4) Augment the column matrix by adding a row corresponding to the new row in the coefficient matrix (row 5). Place in this position the negative of the variable in the row of the column matrix corresponding to the column of negative constant(s) in the coefficient matrix.

$$\begin{bmatrix} s & 1 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 1 \\ 0 & 0 & s & 1 & 0 \\ a_0 & a_1 & a_2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$
$$\begin{Bmatrix} x & -sx & -s^2x & -s^3x & s^2x \end{Bmatrix}$$

Actually, what we have done by following these steps is to add a new node to the system, a node being represented by an amplifier. This shows the power of the matrix notation, and it explains why there are many possible computer connections for each differential equation. Since the matrix may be transformed by a set of elementary operations, many alternate solutions may be obtained. For example, if the node is added to (4.11.2) between $-s^2x$ and s^3x, we obtain

$$\begin{bmatrix} s & 1 & 0 & 0 & 0 \\ 0 & s & 0 & 1 & 0 \\ 0 & 0 & s & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ a_o & a_1 & a_2 & 0 & 1 \end{bmatrix} \qquad (4.11.5)$$
$$\begin{Bmatrix} x & -sx & -s^2x & s^2x & s^3x \end{Bmatrix}$$

The gain ideograph of (4.11.5) is given in Fig. 4.11.2

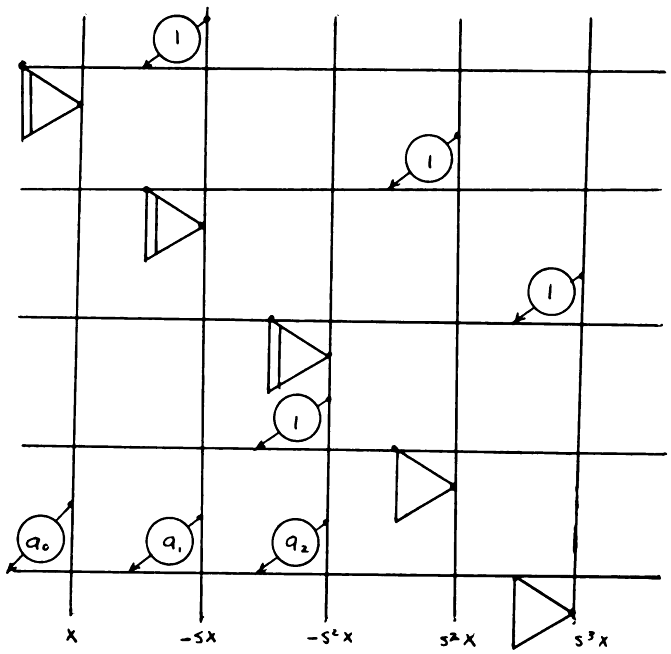


Fig. 4.11.2

The example below will illustrate the procedure of this section when a reflected equation is used as the starting point.

Example 4.11

Given the differential equation

$$2\frac{d^3p}{dt^3} + 10\frac{d^2p}{dt^2} - 5p = 0$$

find the reflected ideograph.

Solution: The equation is written in standard operator notation

$$s^3p + 5s^2p + 0\cdot sp - 2.5p = 0$$

and in reflected matrix form

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 4 & 3 & 2 & 1 \end{array} \\ \begin{bmatrix} 0 & 0 & -1 & s \\ 0 & -1 & s & 0 \\ -1 & s & 0 & 0 \\ 1 & 5 & 0 & -2.5 \end{bmatrix} \\ \left\{ s^3p \quad s^2p \quad sp \quad p \right\} \end{array}$$

Both row-column pairs 1 and 2 have sign sums of -2. Row-columns 3 and 4 have sign sums of -1. Applying EMT 1 to row-column 1 we obtain

$$\begin{bmatrix} 0 & 0 & 1 & s \\ 0 & -1 & s & 0 \\ -1 & s & 0 & 0 \\ 1 & 5 & 0 & 2.5 \end{bmatrix}$$
$$\left\{ s^3p \quad s^2p \quad sp \quad -p \right\}$$

EMT 1 on row-column 3 gives

$$\begin{bmatrix} 0 & 0 & 1 & s \\ 0 & 1 & s & 0 \\ 1 & s & 0 & 0 \\ 1 & -5 & 0 & 2.5 \end{bmatrix}$$
$$\left\{ s^3p \quad -s^2p \quad sp \quad -p \right\}$$

Adding a column and a row

$$\begin{bmatrix} 0 & 0 & 0 & 1 & s \\ 0 & 0 & 1 & s & 0 \\ 0 & 1 & s & 0 & 0 \\ 5 & 1 & 0 & 0 & 2.5 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$
$$\left\{ s^2p \quad s^3p \quad -s^2p \quad sp \quad -p \right\}$$

The ideograph shown in Fig. 4.11.3 follows immediately.

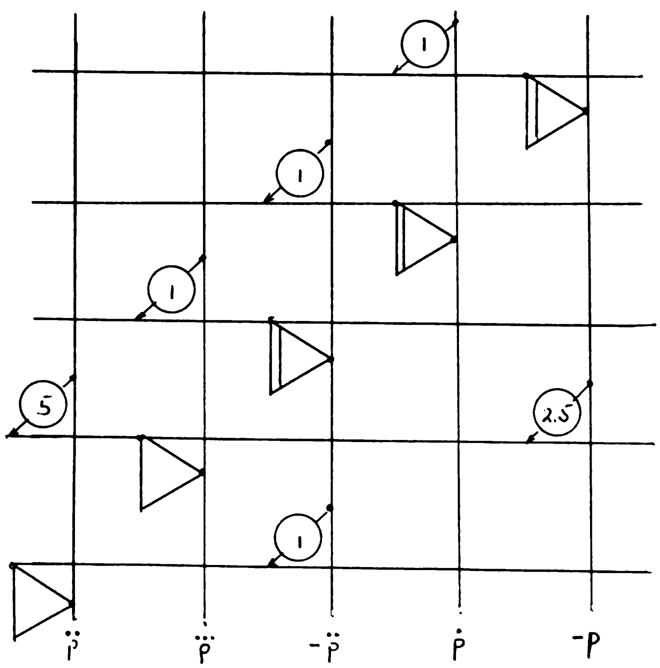


Fig. 4.11.3

4.12 Magnitude Scaling by Matrix Manipulations

Magnitude scaling may be accomplished by applying elementary matrix transformations to the matrix equations. Two more transformations which are needed will first be developed.

Notice that the equation

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ a_0 & a_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \\ s^2x \end{bmatrix} = \begin{bmatrix} s & -\frac{1}{k} & 0 \\ 0 & \frac{1}{k}s & -1 \\ a_0 & \frac{1}{k}a_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ ksx \\ s^2x \end{bmatrix} \qquad (4.12.1)$$

may be shown to be true by matrix multiplication. This leads to EMT 3.

EMT 3

"An element in the jth row of the solution vector may be multiplied by a constant if every element of the jth column of the coefficient matrix is divided by the same constant."

Also notice that an equation is not affected if both sides of the equation are multiplied by the same constant. The equations

$$a_0x + a_1sx + s^2x = 5$$

and

$$ka_0x + ka_1sx + ks^2x = 5k$$

are equivalent. The systems

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ a_0 & a_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \\ s^2x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix},$$

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ ka_0 & ka_1 & k \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \\ s^2x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5k \end{bmatrix},$$

$$\begin{bmatrix} s & -1 & 0 \\ 0 & ks & -k \\ a_o & a_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \\ s^2x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}, \text{ etc.}$$

are also equivalent. This may be stated:

EMT 4

"A row of the coefficient matrix may be multiplied by a constant if the corresponding row of the matrix of constants is multiplied by the same constant."

Magnitude scaling may now be shown with the aid of an example. Example 2.5.1, presented in section 2.5 will be used for this purpose.

The equation was

$$\begin{cases} \ddot{x}(t) + 2\dot{x}(t) + 5x(t) = 0 \\ x(0) = 20 \text{ in} \\ \dot{x}(0) = 20 \text{ in/sec} \end{cases} \qquad (4.12.2)$$

As we recall, the estimated maximum values dictated that the display variables be $\frac{x(t)}{3}$, x/2 and 5x. This means that these desired solutions must appear in the solution vector of the matrix equation rather than the usual variables x, $\dot{x}$, and $\ddot{x}$. This may be accomplished as follows:

1) Write the standard matrix equation for the differential equation

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 5 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \\ s^2x \end{bmatrix} = \mathbf{0} \qquad (4.12.3)$$

2) Introduce the factor of 5 into x, the factor of $\frac{1}{2}$ into sx, and the factor $\frac{1}{3}$ into s^2x by EMT 3.

$$\begin{bmatrix} \frac{s}{5} & -2 & 0 \\ 0 & 2s & -3 \\ 1 & 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5x \\ \frac{1}{2}\, sx \\ \frac{1}{3}\, s^2x \end{bmatrix} = \mathbf{0} \qquad (4.12.4)$$

3) Since the gain ideograph requires that the coefficients of the diagonal elements be equal to unity, EMT 4 is used on each row to secure this. Row 1 is multiplied by 5, row 2 is divided by 2, and row 3 is divided by 3.

$$\begin{bmatrix} s & -10 & 0 \\ 0 & s & -\frac{3}{2} \\ \frac{1}{3} & \frac{4}{3} & 1 \end{bmatrix} \begin{bmatrix} 5x \\ \frac{1}{2} & sx \\ \frac{1}{3} & s^2x \end{bmatrix} = \begin{bmatrix} 5\cdot 0 \\ \frac{0}{2} \\ \frac{0}{3} \end{bmatrix} \qquad (4.12.5)$$

This is the scaled equation. Application of EMT 1 to row-column 2 and addition of a node gives the system

$$\begin{bmatrix} s & 10 & 0 & 0 \\ 0 & s & \frac{3}{2} & 0 \\ \frac{1}{3} & 0 & 1 & \frac{4}{3} \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ -\frac{1}{2}sx \\ \frac{1}{3}s^2x \\ -\frac{1}{2}\, sx \end{bmatrix} = \mathbf{0} \qquad (4.12.6)$$

The ideograph shown in Fig. 4.12.1, which results, is in effect the reflected form of the one obtained by using the highest derivative method.

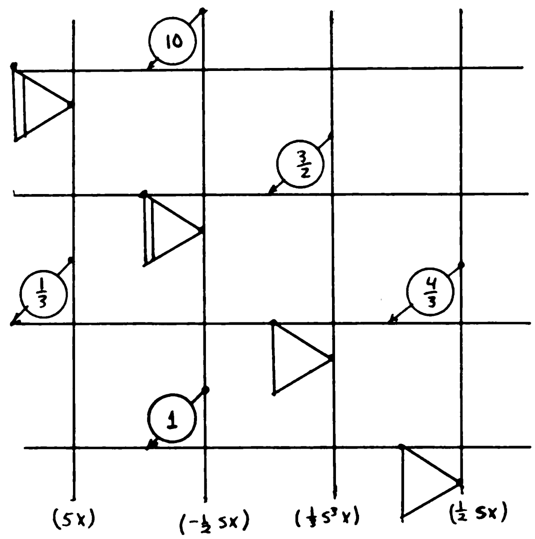


Fig. 4.12.1

Another way to approach magnitude scaling is to write the differential equation in terms of the new display variables as we had done before:

$$\left(\frac{s^2x}{3}\right) + \frac{4}{3}\left(\frac{sx}{2}\right) + \frac{1}{3}(5x) = 0$$

$$(5x_o) = 100 \qquad (4.12.7)$$

$$\left(\frac{\dot{x}(0)}{2}\right) = 10$$

and to use the matrix equation

$$\begin{bmatrix} s & -K_1 & 0 \\ 0 & s & -K_2 \\ \frac{1}{3} & \frac{4}{3} & 1 \end{bmatrix} \cdot \begin{bmatrix} (5x) \\ \frac{1}{2}sx \\ \frac{1}{3}s^2x \end{bmatrix} = 0 \qquad (4.12.8)$$

where K_1 and K_2 are adjusted to satisfy the first two rows of the product matrix. This requires that

$$\begin{cases} 5sx - K_1 \cdot \frac{sx}{2} = 0 \\ \frac{1}{2}s^2x - K_2 \cdot \frac{s^2x}{3} = 0 \end{cases} \qquad (4.12.9)$$

or

$$\begin{cases} K_1 = 10 \\ K_2 = \frac{3}{2} \end{cases} \qquad (4.12.10)$$

It is easy to see that in this case the -1 terms in the standard matrix equation now become constants which depend on the manner in which the display variables are defined. In general, if $-K_j$ is such a constant in the jth row and V_j is the corresponding display variable, then

$$K_j = \frac{sV_j}{V_{j+n}} , \qquad (4.12.13)$$

where n is the number of simultaneous differential equations represented by the matrix equation (in this case n=1). Time scaling may be handled as before, by multiplying the gains of the integrators by an appropriate constant factor. The integrator gains are given by (4.12.13), so that, in general, this equation may be written

$$K_j = \frac{t}{\tau} \frac{sV_j}{V_{j+n}} \qquad (4.12.14)$$

where τ is computer time and t problem time.

4.13 Node Elimination

A technique which is very useful in network analysis and synthesis is the modification of a network by eliminating nodes. We will introduce the technique by showing how a node may be eliminated from the ideograph by operating on the matrix equation.

A node may be eliminated from an ideograph by applying the following rule:

"A node represented by the diagonal element a_{kk} may be eliminated by removing the kth column and the kth row from the matrix equation and by replacing each remaining matrix element a_{ij} in the coefficient matrix with the expression $a_{ij} - \frac{a_{ik}a_{kj}}{a_{kk}}$".

Suppose that in solving the equation

$$a_2S^2X + a_1SX - a_oX = 0 \qquad (4.13.1)$$

we do not wish to obtain S^2X in the analog network.

If the equation is written in standard operator notation, the matrix equation of (4.13.2) results.

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -\frac{a_o}{a_2} & \frac{a_1}{a_2} & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \\ s^2x \end{bmatrix} = 0 \qquad (4.13.2)$$

Since we only wish to display x and sx, the node corresponding to s^2x may be eliminated by deleting row 3 and column 3 and by modifying each remaining matrix element according to the node elimination rule. This gives:

$$\begin{bmatrix} s - (0)\cdot(-\frac{a_o}{a_2})/1 & -1-(0)\cdot(\frac{a_1}{a_2})/1 & 0 \\ 0 - (-1)\cdot(-\frac{a_o}{a_2})/1 & s-(-1)\cdot(\frac{a_1}{a_2})/1 & -1 \\ -\frac{a_o}{a_2} & \frac{a_1}{a_2} & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \\ s^2x \end{bmatrix} = 0 \qquad (4.13.3)$$

or

$$\begin{bmatrix} s & -1 \\ \frac{a_o}{a_2} & s + \frac{a_1}{a_2} \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \end{bmatrix} = 0 \qquad (4.13.4)$$

Application of EMT 1 to row-column 1 gives

$$\begin{bmatrix} s & 1 \\ \frac{a_o}{a_2} & s + \frac{a_1}{a_2} \end{bmatrix} \cdot \begin{bmatrix} -x \\ sx \end{bmatrix} = 0 \qquad (4.13.5)$$

This equation may be verified by performing the matrix multiplication:

$$\begin{cases} -sx + sx = 0 \\ -\frac{a_o}{a_2}x + (s + \frac{a_1}{a_2})\ sx = -\frac{a_o}{a_2}\ x + s^2x + \frac{a_1}{a_2}\ sx = 0 \end{cases} \qquad (4.13.6)$$

which reproduces the given differential equation. Elimination of node 3 yields an unfamiliar term $(s + \frac{a_1}{a_2})$ as one of the diagonal elements. It can be shown that this term may be interpreted as adding an extra gain equal to $\frac{a_1}{a_2}$ across amplifier 1 as shown in the ideograph of Fig. 4.13.1.

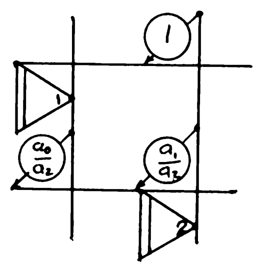


Fig. 4.13.1

This indicates that the output of integrator No. 2 must be connected to its input through a gain of a_1/a_2.

We conclude that elimination of node 3 has produced a simpler circuit which is preferable whenever the second derivative is not needed.

Node elimination, however, has its practical limitations when applied to analog networks. This is because diagonal terms of the form (s + k) may be interpreted as was done above, but terms of the form (s - k) may not. Although in some cases it is possible to change the sign of the constant term by elementary matrix transformations, in some cases this may not be possible. For example, in the system.

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ a_o & -a_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \\ s^2x \end{bmatrix} = \mathbf{0} \qquad (11.13.7)$$

Elimination of node 3 gives

$$\begin{bmatrix} s & -1 \\ a_o & s-a_1 \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \end{bmatrix} = \mathbf{0} \qquad (11.13.8)$$

The two negative constants may be eliminated by adding a node

$$\begin{bmatrix} s & 0 & 1 \\ a_o & s & a_1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \\ -sx \end{bmatrix} = \mathbf{0}, \qquad (11.13.9)$$

but the procedure gives an ideograph with as many amplifiers as would be obtained if the second derivative was retained.

This example has shown two things which we had not encountered before. One is the fact that when a node is added, all of the negative constants in the same column may be eliminated by moving them to the new column. Secondly, the negative portion of a term of the form (s - k) may be treated as any single negative constant and may also be moved, by itself, to the new column.

4.14 Other Elementary Matrix Transformations

Two additional elementary matrix transformations which are very useful in simplifying analog networks will now be given. The first transformation is stated as follows:

EMT 5

"A row of the coefficient matrix may be multiplied by a constant and added to any other row"

Example 11.14.1

The equation

$$-a_o x + a_1 sx + a_2 s^2 x = 0 \qquad (4.14.1)$$

is satisfied by the matrix equation

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ \frac{a_o}{a_2} & \frac{a_1}{a_2} & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \\ s^2x \end{bmatrix} = \mathbf{0} \qquad (4.14.2)$$

as well as by the equation

$$\begin{bmatrix} s & -1 & 0 \\ -\frac{a_o}{a_2} & s+\frac{a_1}{s_2} & 0 \\ -\frac{a_o}{a_2} & \frac{a_1}{a_2} & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \\ s^2x \end{bmatrix} = \mathbf{0} \qquad (4.14.3)$$

which results when row 3 is multiplied by 1 and added to row 2. This may be verified by performing the matrix multiplication.

Another elementary matrix transformation may be discovered if EMT 1 is applied on row-column 1 of (4.14.3) to obtain

$$\begin{bmatrix} s & 1 & 0 \\ \frac{a_o}{a_2} & s+\frac{a_1}{a_2} & 0 \\ \frac{a_o}{a_2} & \frac{a_1}{a_2} & 1 \end{bmatrix} \cdot \begin{bmatrix} -x \\ sx \\ s^2x \end{bmatrix} = \mathbf{0} \qquad (4.14.4)$$

If (4.14.4) is now compared to (4.13.5), we notice that node 3 is actually not needed, since (4.13.5) satisfies the given differential equation and requires one less amplifier.

Another way to reach the same conclusion is to notice that if node 3 is eliminated from (4.14.4) the remaining entries will not be affected, since all the elements of column 3 are zero except the diagonal. This leads to the concept of a redundant node, and to the following EMT.

EMT 6

> "Whenever all of the off-diagonal entries in a column are zero, the corresponding node is redundant and may be eliminated without affecting the rest of the matrix."

EMT 6 suggests a method for eliminating nodes. The method consists on using EMT 5 to zero out all the off-diagonal column entries corresponding to a node, and then to apply EMT 6 to eliminate the node.

4.15 Ordinary Differential Equations with Time-Dependent Forcing Functions.

Consider the differential equation

$$(a_os^o + a_1s^1 + a_2s^2 + \ldots + a_ns^n)\, x = f(t), \qquad (4.15.1)$$

where the right side of the equation is a function of the independent variable t.

If the equation is written in the form

$$\left(\frac{a_o}{a_n} s^o + \frac{a_1}{a_n} s^1 + \frac{a_2}{a_n} s^2 + \dots + s^n \right) x - \frac{f(t)}{a_n} = 0 \qquad (4.15.2)$$

We find that the matrix equation

$$\begin{bmatrix} s & -1 & 0 \dots & 0 & 0 & 0 \\ 0 & s & -1 \dots & 0 & 0 & 0 \\ 0 & 0 & s \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ 0 & 0 & 0 & s & -1 & 0 \\ \frac{a_o}{a_n} & \frac{a_1}{a_n} & \frac{a_2}{a_n} & \frac{a_{n-1}}{a_n} & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ sx \\ s^2x \\ \cdot \\ \cdot \\ \cdot \\ s^{n-1}x \\ s^n x \\ \frac{1}{a_n} f(t) \end{bmatrix} = 0 \qquad (4.15.3)$$

will satisfy the given differential equation. This matrix equation is somewhat unusual because all the elements in one of the rows are zero, including the diagonal. We will find, however, that this matrix will behave properly when applied any of the elementary matrix tranformations which we have used in connection with other matrices.

Notice that node 6 is not redundant, since its diagonal element is not equal to 1.

Example 4.15

The equation

$$a_o y + a_1 \frac{dy}{dt} + a_2 \frac{d^2y}{dt^2} = f(t) \qquad (4.15.4)$$

may be written in the form

$$\begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ \frac{a_o}{a_2} & \frac{a_1}{a_2} & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (4.15.5)$$

$$\left\{ y \quad sy \quad s^2y \quad \frac{f(t)}{a_2} \right\}$$

EMT 1 applied to row-columns 2 and 4 yields

$$\begin{bmatrix} s & 1 & 0 & 0 \\ 0 & s & 1 & 0 \\ \frac{a_o}{a_2} & -\frac{a_1}{a_2} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.15.6}$$

$$\left\{ y \quad -sy \quad s^2y \quad -\frac{f(t)}{a_2} \right\}$$

The negative entry may be eliminated by adding a node to obtain

$$\begin{bmatrix} s & 1 & 0 & 0 & 0 \\ 0 & s & 1 & 0 & 0 \\ \frac{a_o}{a_2} & 0 & 1 & 1 & \frac{a_1}{a_2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{4.15.7}$$

$$\left\{ y \quad -sy \quad s^2y - \frac{f(t)}{a_2} \quad sy \right\}$$

This ideograph is given in Fig. 4.15.1

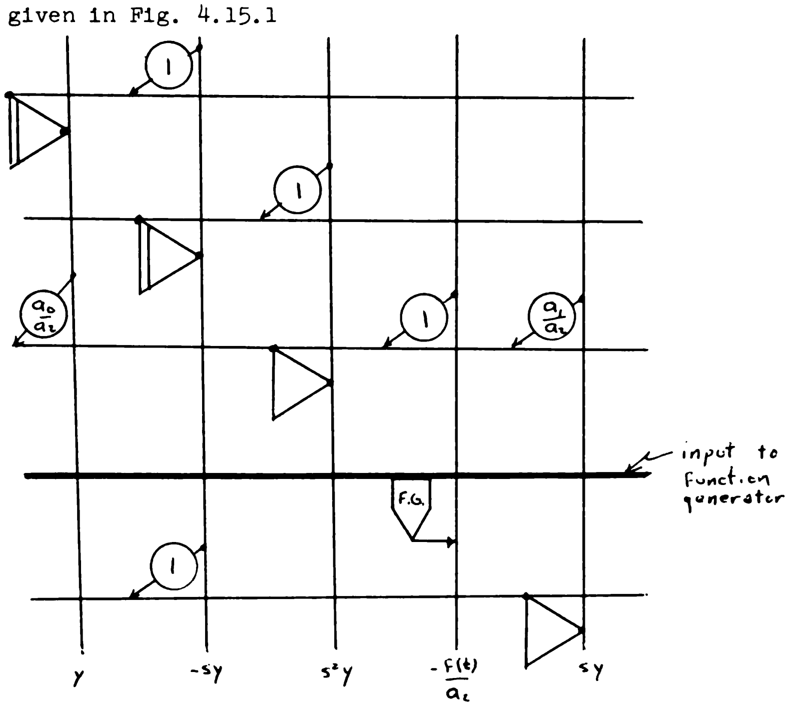


Fig. 4.15.1

Notice that the zero diagonal in row 4 indicates that node four does not contain an amplifier. Also notice that since the term $-f(t)/a_2$ appears as an entry in the solution vector, it will appear as an output and will be represented as a vertical line in the ideograph. The actual function $-f(t)/a_2$ is the output of a function generator which may be placed in the location usually occupied by the amplifier. It is convenient to think of the horizontal line representing row 4 as the input line of the function generator. For example, if the forcing function in equation (4.15.4) is a function of y and s^2y rather than t, the connection shown in Fig. 4.15.2 between the output of amplifiers 1 and 3 and the function generator input line would indicate the arguments of the function.

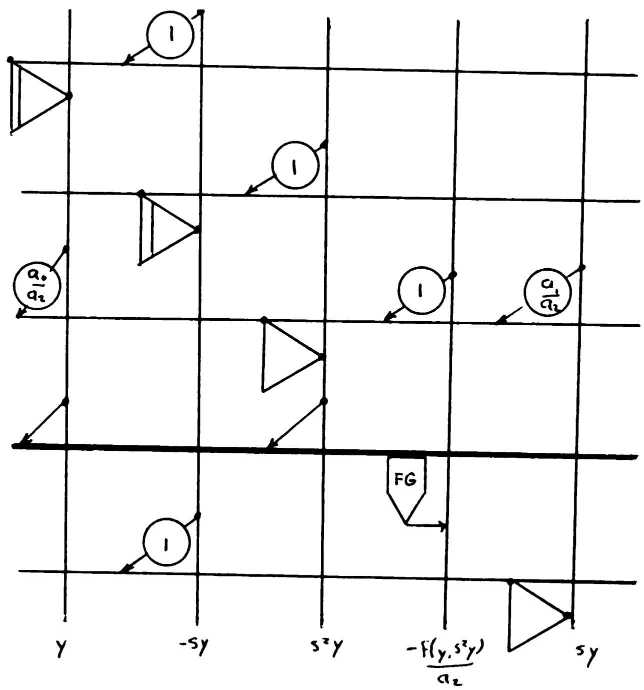


Fig. 4.15.2

4.16 Equation with Variable Coefficients

In the differential equation

$$y + (1-y)\frac{dy}{dt} + \frac{d^2y}{dt^2} = 0 \qquad (4.16.1)$$

the coefficient of the first derivative $\frac{dy}{dt}$ is a function of the dependent variable y. If we write the equation in the form

$$\left(y - \frac{dy}{dt} + \frac{d^2y}{dt^2}\right) = y\frac{dy}{dt} \qquad (4.16.2)$$

and define $y \frac{dy}{dt} = f(y, sy)$ then the equation may be assumed to be an ordinary differential equation with a forcing function which depends on y and $\frac{dy}{dt}$. The matrix equation has the form of (4.15.3) and is given by

$$\begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ sy \\ s^2y \\ ysy \end{bmatrix} = \mathbf{0} \quad , \qquad (4.16.3)$$

which yields the transformed matrix equation

$$\begin{bmatrix} s & 1 & 0 & 0 \\ 0 & s & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ -sy \\ s^2y \\ -ysy \end{bmatrix} = \mathbf{0} \quad , \qquad (4.16.4)$$

and the ideograph shown in Fig. 4.16.1

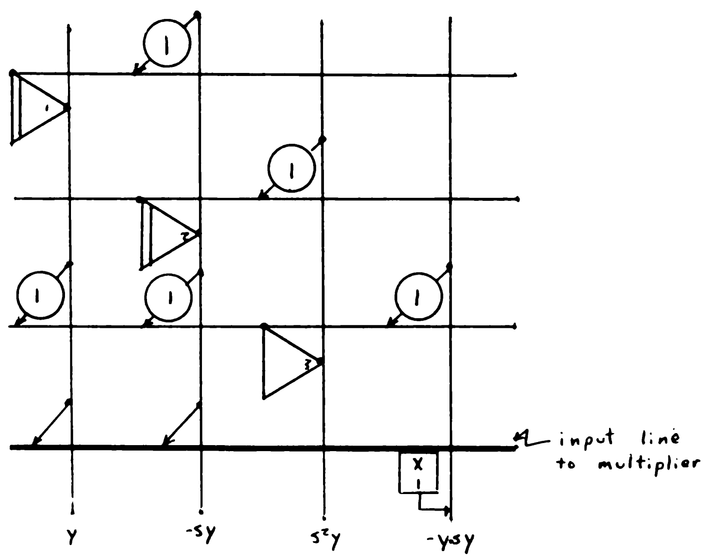


Fig. 4.16.1

4.17 Simultaneous Differential Equations

The techniques of the previous paragraphs may be extended to the solution of simultaneous differential equations. For example, let us consider the differential equations

$$\begin{cases} \ddot{x}(t) + 2a\,\dot{y}(t) + b^2\,x(t) = 0 \\ \ddot{y}(t) - 2a\,\dot{x}(t) + b^2\,y(t) = 0. \end{cases} \qquad (4.17.1)$$

These equations may be written in standard operator notation

$$\begin{cases} s^2x + 0\cdot s^2y + 0\cdot sx + 2a\cdot sy + b^2x + 0\cdot y = 0 \\ 0\cdot s^2x + s^2y - 2a\cdot sx + 0\cdot sy + 0\cdot x + b^2\cdot y = 0 \end{cases} \qquad (4.17.2)$$

in which each equation contains every derivative. Then we may write the equation

$$\begin{bmatrix} s^2 & 0 \\ 0 & s^2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 & 2a\cdot s \\ -2a\cdot s & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b^2 & 0 \\ 0 & b^2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (4.17.3)$$

Factoring out the solution vector

$$\left\{ \begin{bmatrix} s^2 & 0 \\ 0 & s^2 \end{bmatrix} + \begin{bmatrix} 0 & 2a\cdot s \\ 2a\cdot s & 0 \end{bmatrix} + \begin{bmatrix} b^2 & 0 \\ 0 & b^2 \end{bmatrix} \right\} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (4.17.4)$$

Factoring out the s^2 operator from the first matrix and the s operator from the second we obtain

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 2a \\ -2a & 0 \end{bmatrix} s + \begin{bmatrix} b^2 & 0 \\ 0 & b^2 \end{bmatrix} \right\} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (4.17.5)$$

This is an equation in which the coefficients of the operators and the dependent variable turn out to be matrices rather than single numbers. After a little practice, the matrix equation may be obtained by inspection from the original equations.

Equations (4.17.5) may then be written in standard matrix form

$$\left[\begin{array}{cc|cc|cc} s & 0 & -1 & 0 & 0 & 0 \\ 0 & s & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & s & 0 & -1 & 0 \\ 0 & 0 & 0 & s & 0 & -1 \\ \hline b^2 & 0 & 0 & 2a & 1 & 0 \\ 0 & b^2 & -2a & 0 & 0 & 1 \end{array}\right] \qquad (4.17.6)$$

$$\{x \quad y \quad sx \quad sy \quad s^2x \quad s^2y\}$$

Notice that this equation may be written directly from the given equations. The derivatives of x and y increase from left to right as before, with derivatives of the same order following each other.

That equation (4.17.6) does not contain any contradictions and reproduces the two given differential equations may be shown by preforming the indicated multiplication. The dashed lines are used to emphasize the fact that this matrix form is very similar to the one we used for a single differential equation, except in that the matrices

$$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}, \qquad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are used in place of s, -1, and 0, and that the original coefficients a_0, a_1, ... , a_{n-1}, 1 are now the matrix ooefficients of (4.17.5).

As in the case of a single equation, (4.17.6) is one of the many forms in which the simultaneous equations may be written. All of the elementary matrix transformations which applied to the matrix equations in the previous sections also apply to (4.17.6).

If a "sign map" is prepared for this system, it will be found that a minimal amplifier system is given by

$$\begin{bmatrix} s & 0 & 1 & 0 & 0 & 0 \\ 0 & s & 0 & 1 & 0 & 0 \\ 0 & 0 & s & 0 & 1 & 0 \\ 0 & 0 & 0 & s & 0 & 1 \\ b^2 & 0 & 0 & -2a & 1 & 0 \\ 0 & b^2 & 2a & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ -sx \\ -sy \\ s^2x \\ s^2y \end{bmatrix} = 0 \qquad (4.17.7)$$

Addition of a node produces the ideograph equation

$$\begin{bmatrix} s & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & s & 0 & 1 & 0 \\ b^2 & 0 & 0 & 0 & 1 & 0 & 2a \\ 0 & b^2 & 2a & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad (4.17.8)$$

$$\left\{ x \quad y \quad -sx \quad -sy \quad s^2x \quad s^2y \quad sy \right\}$$

Another example will further illustrate the method.

Example 4.17.1

Solve the differential equations

$$\begin{cases} \ddot{y}(t) + 234\, y(t) - 1296\, \theta = 0 \\ 8\ddot{\theta}(t) - 54\, y(t) + 5616\, \theta = 0 \end{cases} \qquad (4.17.9)$$

with initial conditions

$$\begin{cases} y(0) = 100 \qquad \dot{y}(0) = 0 \\ \theta(0) = 1.47 \qquad \dot{\theta}(0) = 0 \end{cases} \qquad (4.17.10)$$

and the display variables

$$\left(\frac{\ddot{y}}{250}\right), \left(\frac{\dot{y}}{20}\right), \quad y$$

$$\left(\frac{\ddot{\theta}}{5}\right), \quad (2\dot{\theta}), \quad (50\theta)$$

The equation is written in standard operator notation

$$\left\{\begin{array}{l} s^2y + 0\cdot s^2\theta + 0\cdot sy + 0\cdot s\theta + 234y - 1296\theta = 0 \\ 0\cdot s^2y + \quad s^2\theta + 0\cdot sy + 0\cdot s\theta - 6.75y + 702\theta = 0 \end{array}\right. \qquad (4.17.11)$$

and in matrix notation

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot s + \begin{bmatrix} 234 & -1296 \\ -6.75 & 702 \end{bmatrix} \right\} \cdot \begin{bmatrix} y \\ \theta \end{bmatrix} = 0 \qquad (4.17.12)$$

The standard matrix equation is

$$\begin{bmatrix} s & 0 & -1 & 0 & 0 & 0 \\ 0 & s & 0 & -1 & 0 & 0 \\ 0 & 0 & s & 0 & -1 & 0 \\ 0 & 0 & 0 & s & 0 & -1 \\ 234 & -1296 & 0 & 0 & 1 & 0 \\ -6.75 & 702 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ \theta \\ sy \\ s\theta \\ s^2y \\ s^2\theta \end{bmatrix} = 0 \qquad (4.17.13)$$

The display variables are introduced into the solution vector by application of EMT 3

$$\begin{bmatrix} s & 0 & -20 & 0 & 0 & 0 \\ 0 & \frac{s}{50} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 20s & 0 & -250 & 0 \\ 0 & 0 & 0 & \frac{s}{2} & 0 & -5 \\ 234 & -25.92 & 0 & 0 & 250 & 0 \\ -6.75 & 14.04 & 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} y \\ 50\theta \\ \frac{sy}{20} \\ 2s\theta \\ \frac{s^2y}{250} \\ \frac{s^2\theta}{5} \end{bmatrix} = 0 \qquad (4.17.14)$$

Now each row of the coefficient matrix may be multiplied or divided by a constant by using EMT 4 so that all diagonal elements become unity. This gives.

$$\begin{bmatrix} s & 0 & -20 & 0 & 0 & 0 \\ 0 & s & 0 & -25 & 0 & 0 \\ 0 & 0 & s & 0 & -12.5 & 0 \\ 0 & 0 & 0 & s & 0 & -10 \\ .936 & -.104 & 0 & 0 & 1 & 0 \\ -1.35 & 2.808 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ 50\theta \\ \frac{sy}{20} \\ 2\cdot s\theta \\ \frac{s^2y}{250} \\ \frac{s^2\theta}{5} \end{bmatrix} = 0 \qquad (4.17.15)$$

A sign map would indicate that row-columns 2, 3, and 6 should be changed in sign. This yields the ideograph equation

$$\begin{bmatrix} s & 0 & 20 & 0 & 0 & 0 \\ 0 & s & 0 & 25 & 0 & 0 \\ 0 & 0 & s & 0 & 12.5 & 0 \\ 0 & 0 & 0 & s & 0 & 10 \\ .936 & .104 & 0 & 0 & 1 & 0 \\ 1.35 & 2.81 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.17.16}$$

$$\left\{ y \quad -50\theta \quad -\frac{sy}{20} \quad 2s\theta \quad \frac{s^2y}{250} \quad -\frac{s^2\theta}{5} \right\}$$

Notice that every row containing s in (4.17.16) has a gain which is of the order of ten times larger than unity. This is to be expected since the estimated undamped frequencies for the equations may be found to be $\omega_y \approx 15$ rad/sec and $\omega_\theta \approx 30$ rad/sec. This suggests that all of the integrator gains be divided by 10 to slow down the solution to a frequency range at which the analog components respond more accurately.

An alternative procedure for solving this example is to write the given differential equations in scaled form

$$\begin{cases} \left(\frac{\ddot{y}}{250}\right) + .936(y) - .104(50\theta) = 0 \\ \left(\frac{\ddot{\theta}}{5}\right) - 1.35(y) + 2.81\,(50\theta) = 0 \end{cases}$$

$$\begin{cases} y(0) = 100 & \left(\frac{\dot{y}(0)}{20}\right) = 0 \\ 50\theta(0) = 73.5 & 2\dot{\theta}(0) = 0. \end{cases}$$

Equation 4.17.15 then follows directly if equation (4.12.13), with n = 2, is used in obtaining the K constants. This gives

$$K_1 = \frac{sV_1}{V_3} = \frac{s(y)}{\frac{sy}{20}} = 20$$

$$K_2 = \frac{sV_2}{V_4} = \frac{s(50\theta)}{2\,s\theta} = 25$$

$$K_3 = \frac{sV_3}{V_5} = \frac{s(sy/20)}{\frac{s^2y}{250}} = 12.5$$

$$K_4 = \frac{sV_4}{V_6} = \frac{s(2\,s\theta)}{\frac{s^2\theta}{5}} = 10$$

4.1 Problems

In the expression $\mathbf{A} \cdot \mathbf{B}$ we may say that $\mathbf{B}$ is pre-multiplied by $\mathbf{A}$ and that $\mathbf{A}$ is post-multiplied by $\mathbf{B}$. Show that in general a different result is obtained if $\mathbf{A}$ is pre-multiplied by $\mathbf{B}$ than if $\mathbf{A}$ is post-multiplied by $\mathbf{B}$.

4.2 If $\mathbf{A} \cdot \mathbf{B}$ exists, does $\mathbf{B} \cdot \mathbf{A}$ exist?

4.3 Find the product P·Q of the matrices below

$$P = \begin{bmatrix} a & -1 & 0 \\ 0 & a & -1 \\ 3 & -2 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} a \\ a^2 \\ a^3 \end{bmatrix}$$

4.4 Find R · S and S · R if R and S are given by

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K \end{bmatrix} \qquad S = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

4.5 Reoeat problem 4.4 with

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and write down a rule in regard to pre-multiplying or post-multiplying a matrix by another matrix containing unity in all diagonals except one.

4.6 Show that $\mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C} = (\mathbf{A} + \mathbf{B}) \cdot \mathbf{C}$

that is, that the distributive law of multiplication applies when the variables are matrices.

4.7 Solve the following problems by the matrix approach:

a) Problem 2.1

b) Problem 2.2

c) Problem 2.6

d) Problem 2.7

e) Problem 2.8

4.8 Find the gain ideograph corresponding to the equations:

$$\begin{cases} 400x(t) + 1500y(t) - 300\dot{y}(t) + 20\ddot{x}(t) = 0 \\ 1200x(t) + 2500y(t) + 200\dot{x}(t) - 100\dot{y}(t) + 50\ddot{y}(t) = 0 \end{cases}$$

and the display variables:

$$2x(t) \quad , \quad 5y(t) \quad , \quad \frac{1}{2}\dot{x}(t) \quad , \quad \dot{y}(t) \; , \quad \frac{1}{10}\ddot{x}(t) \quad , \quad \frac{1}{10}\ddot{y}(t) \quad .$$

4.9 Obtain an ideograph for Problem 4.7a in which the third derivative $\dddot{x}(t)$ is not displayed.

4.10 Eliminate $\ddot{s}(t)$ from the ideograph of Problem 4.7b.

4.11 Find the integrator gains k_1, k_2, k_3, and k_4 in the scaled system below and recommend a change in time scale if it would seem required

$$\begin{bmatrix} s & 0 & -K_1 & 0 & 0 & 0 \\ 0 & s & 0 & -K_2 & 0 & 0 \\ 0 & 0 & s & 0 & -K_3 & 0 \\ 0 & 0 & 0 & s & 0 & -K_4 \\ 1.25 & -.26 & 0 & -1 & 1 & 0 \\ -2.5 & 1.8 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 100x \\ 500y \\ 10sx \\ 50sy \\ \frac{s^2x}{2} \\ s^2y \end{bmatrix} = 0$$

4.12 Obtain the minimal amplifier circuit for the matrix equation of Problem 4.11 by using the sign map technique. Add enough nodes to produce an ideograph which uses single-output amplifiers.

Answer to Problems

Exercise 1.1

a) -15 volts. b) +5v. c) .15 d) 5.5 e) 20v. f) $-3E_1$ g) 5

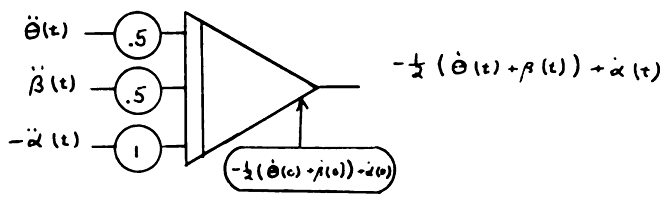
h) .3 i) 10t+30 j) $\frac{1}{2}t^2-5$ k) -y(t) l) $-50t-\dot{y}(t)$ m) $-\ddot{y}(t)$

n) .1 o) initial = 5 gain=.5 p) top gain = 1, bottom gain = A, initial = -Aw(o)

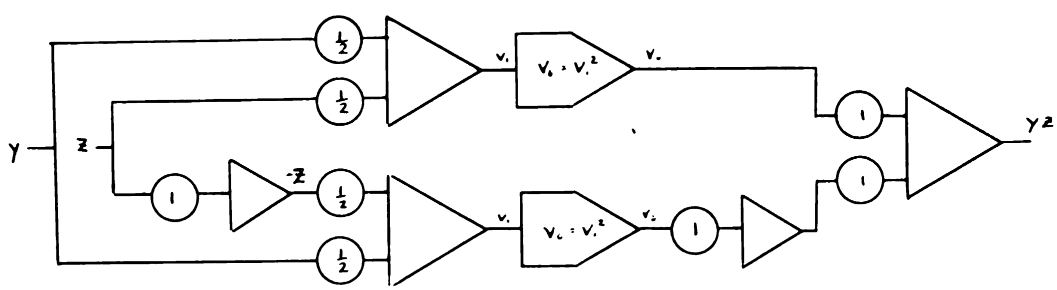
q) $\frac{1}{2}$, 1, $\frac{3}{2}$, 2

Chapter 1

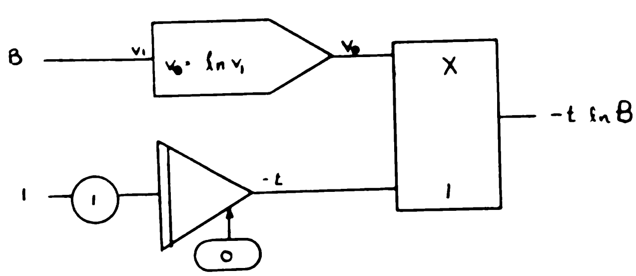
1.1



1.2

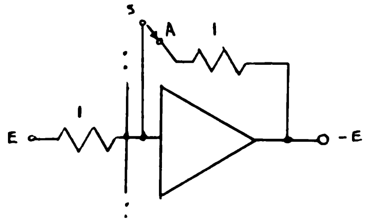


1.3

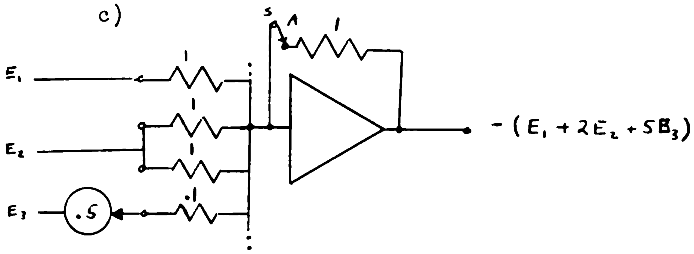


1.5

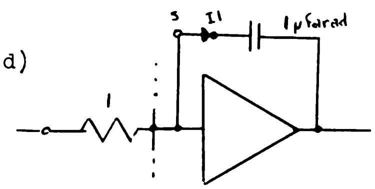
a)



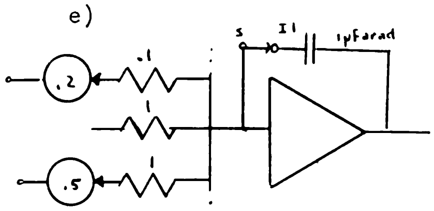
c)



d)

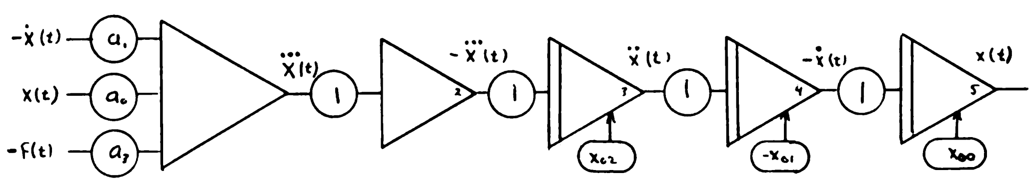


e)

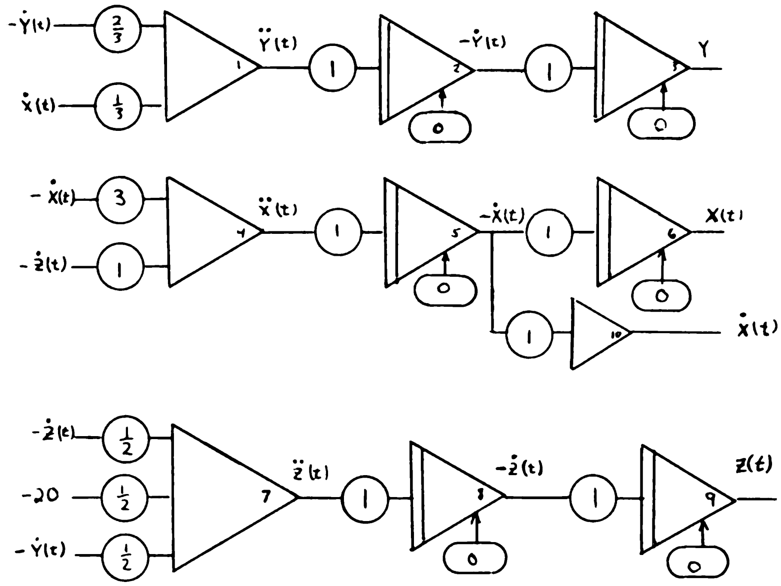


<u>Chapter 2</u>

2.1

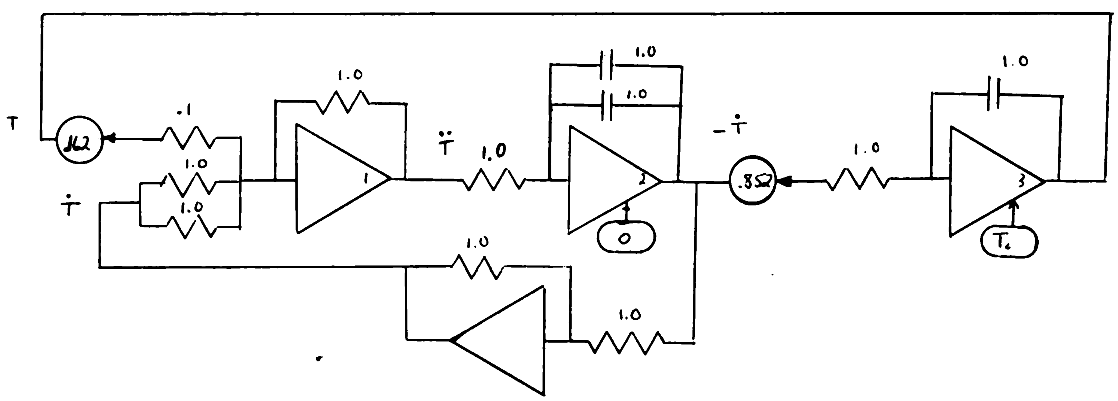


2.2



If only $\dot{x}(t)$, $\dot{y}(t)$ and $\dot{z}(t)$ are required, amplifiers 3, 6, and 9 may be eliminated.

2.3



2.4 a)

$$\ddot{P}(t) - 4\dot{P}(t) + 200\,P(t) = 10f(t)$$

$$P(o) = -50, \quad \dot{P}(o) = 0$$

b) Assuming that $\dot{R}(o) = 50$ and $-\frac{1}{2}R(o) = 5$, the output of amplifier 1 is given by

$$\dot{R}(t) = -\int \left[-\frac{1}{2}R(t) + \frac{1}{3}\dot{R}(t) - 12 \right] dt$$

differentiating both sides of the equation

$$\ddot{R}(t) = \frac{1}{2}R(t) - \frac{1}{3}\dot{R}(t) + 12$$

answer:

$$\ddot{R}(t) + \frac{1}{3}\dot{R}(t) - \frac{1}{2}R(t) = 12$$

$$R(o) = -10, \quad \dot{R}(o) = 50$$

2.5 a)

Amplifier	Output
1	+100
2	0
3	-100

b) 1 and 4

c) Amplifier 1 would try to reach +200 volts and would become overloaded.

Chapter 4

4.7a

$$\begin{bmatrix} s & 1 & 0 & 0 & 0 & 0 \\ 0 & s & 1 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 & 1 \\ a_o & a_1 & 0 & 1 & a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

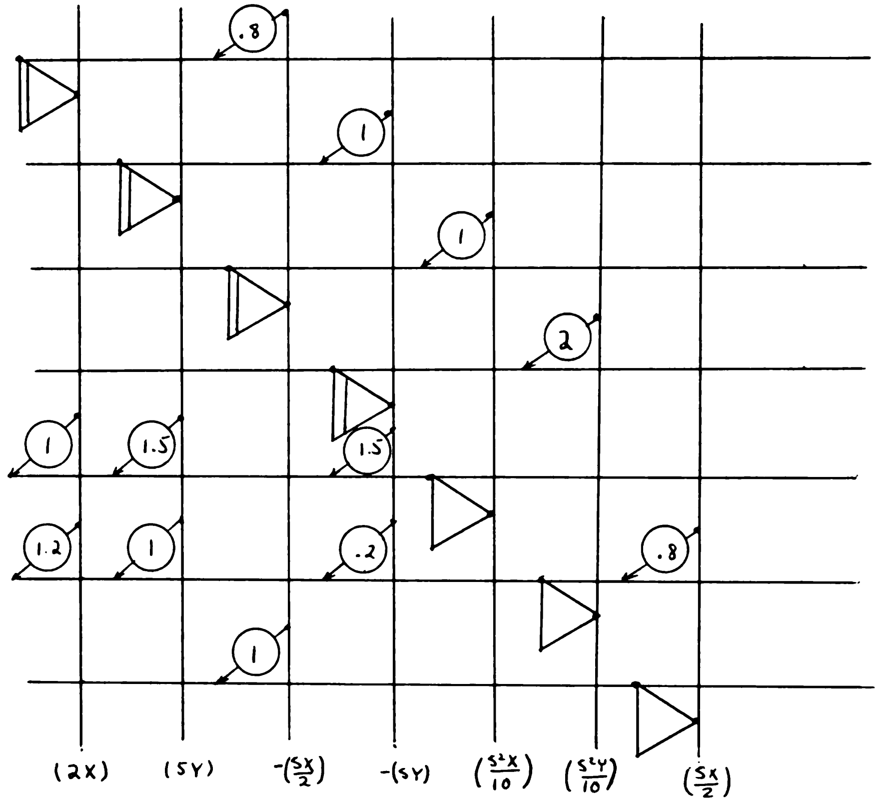
$$\left\{ x \quad -sx \quad s^2x \quad s^3x \quad -f(t) \quad -s^3x \right\}$$

4.7b

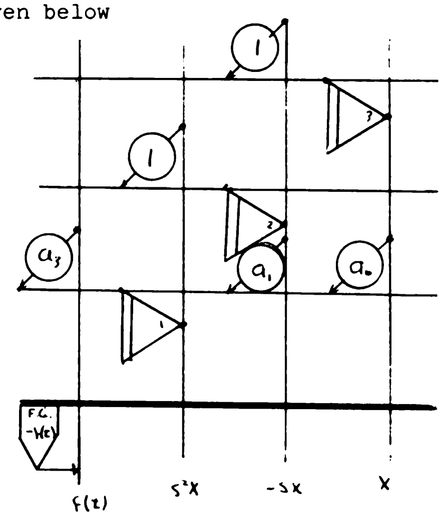
$$\begin{bmatrix} s & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & s & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left\{ y \quad x \quad z \quad -sy \quad -sx \quad -sz \quad s^2y \quad s^2x \quad s^2z \quad -20 \quad sx \right\}$$

4.8 The following ideograph assumes a time slow-down of 5:1



4.9 The reflected gain dieograph is given below



4.11 $K_1 = 10$, $K_2 = 10$, $K_3 = 20$, $K_4 = 50$

A scale change may be necessary depending on the particular computer used.

4.12 EMT 1 on row-columns 1, 4, and 5 produces the minimal circuit

References

1. "Matrix Programming of Electronic Analog Computers," R. E. Horn and P. M. Honnell, AIEE Transactions, Part I, (Communication and Electronics, P.P. 420 - 426, September 1958)

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3. "Analogue Computer Synthesis and Error Matrices," P. M. Honnell, R. E. Horn, AIEE Transactions, Vol. 75, pt. I, Mar. 1956, P.P. 26-32

4. "Electronic Network Synthesis of Linear Algebraic Matrix Equations," R. E. Horn, P. M. Honnell, Transactions of the AIEE, Vol. 78, pt. I, Jan. 1960, P.P. 1028 - 1032.