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Documentation for Analog Computer

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Design and Analysis of an Analog Computer: From Concept to Final System

Abstract:

This paper contains the result of research into analog computation, referring to the computational method that involves constructing an electronic system that is analogous to the system in the problem at hand, utilizing analog voltages to represent the quantities in this problem, and operating on these quantities with operational amplifier circuits that can compute summation, multiplication (by either a constant of another value set by a second voltage), and integration with respect to time. It should be noticed that differences, division, and even arbitrary functions can be implemented or approximated where necessary under this framework. This paper constructs a system based on the above criteria and documents its strengths and weaknesses as a computational paradigm. Finally, conclusions are drawn regarding the viability of the construction of a large scale analog computer.

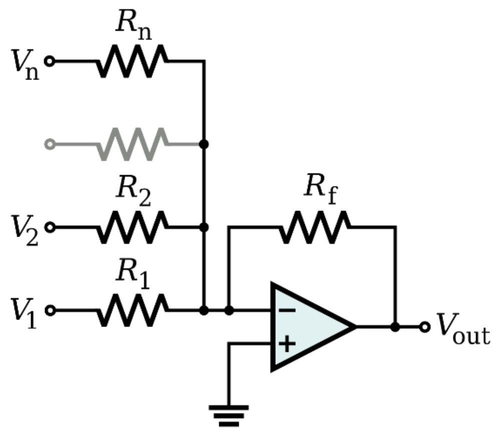
Introduction: Basic Concepts of Analog Computation

As introduced in ECE 2040 or equivalent courses, Kirchhoff's laws can be utilized to analyze circuits through node voltage analysis or mesh current analysis [1]. This theory can be utilized with operational amplifier integrated circuits to construct useful electronic circuits that can model certain operations and mathematical problems. In fact, analog circuits can be constructed that model most mathematical operations, including multiplication, addition, and integration. These three circuits were critical to the design of an analog computer, and thus they are detailed below.

The summation amplifier

The summation amplifier, or summing amplifier, takes a sum of the voltages at its inputs and scales the result. This works rather simply: essentially an operational amplifier has two nodes as its inputs. The third pin, or output of the amplifier "does whatever is necessary" to make the voltage on the two input nodes equal [2] (or, alternately, to make the voltage difference between the two pins equal to zero [1]). This fact, with KCL and KVL, constitutes the basis of the technique of analyzing operational amplifiers – at least when we assume the device is an "ideal operational amp". In this instance, such an assumption is valid to a first order. Furthermore, it should be noticed that an ideal operational amplifier has inputs that have an infinite input resistance and an output that has a thevenin equivalent resistance that is exactly zero. Real devices are unable to reach this, but for the sake of this project it may be assumed that these conditions hold.

The schematic of a summation amplifier is the following diagram.



Summation Amplifier – Fig.1.

The operational amplifier with its two inputs is the bottom right triangular shape. The "-" input is the "inverting" input and the "+" input is the "noninverting" input, respectively. The circuit's output makes the inputs have an equal voltage. Thus the "-" input is at ground potential [1]. From these details, its behavior follows the equations below (Eq.1. and Eq.2.).

$$\frac{V_1}{R_1}+\frac{V_2}{R_2}+\frac{V_3}{R_3}+\cdots+\frac{V_n}{R_n}= I_{Into\ Junction} = I_{through\ Rf} \text{ Eq.1.}$$

Thus, the voltage at the inverting input must be held at ground potential. This implies that the output voltage of this system obeys the equation below, Eq.2.

$$0-R_f(\frac{V_1}{R_1}+\frac{V_2}{R_2}+\frac{V_3}{R_3}+\cdots+\frac{V_n}{R_n}) = -R_f(I_{Into\ Junction}) = -R_f(I_{through\ Rf}) = V_{out} \text{ Eq.2.}$$

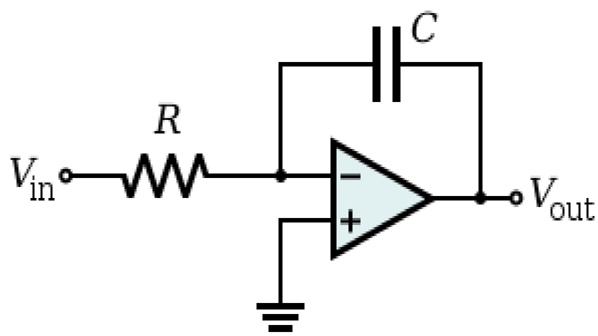
Since the output voltage term, V_{out}, remains at a potential that is *below* ground potential (zero volts), it must be set such that the above equations hold. In this way, the output voltage of this circuit is established such that the output is the negated summation of the voltages on the inputs, scaled by a constant factor that is proportional to R_f. Thus, an amplifier that multiplies by a constant can be created, by using this circuit with only a single input.

It should be noticed that the reason that the "+" input (noninverting input) is connected to ground is not arbitrary. The noninverting summation amplifier is more complicated and is likely to suffer from low input impedance. This limits the utility of this circuit to the extent that it is not considered or utilized in this project. Also, since the "differentiator" circuit is prone to the generation of noise and other adverse effects, it is not included for the same reason [1,2].

If analog electronics could only perform addition and multiplication, there would not be any reason to write this paper. Digital electronics can perform such operations with mush more versatility and speed. The fascinating thing is that this is not the case, however. Analog electronic circuits can be constructed to model differential equations, using a circuit that can take the time integral of its inputs. This circuit, perhaps the most important one in this paper, makes integration almost a trivial operation for an analog computer. In fact, it only takes a single resistor, capacitor, and operational amplifier to perform time integration [1]. This fact, along with the realization that many useful dynamic systems are modeled by differential equations, is what lead me to work on analog computing and this project. Thus, we cover the integrator circuit on the following page.

The Integrator

Utilizing the assumptions of the "ideal operational amplifier", we may analyze the integrator circuit below. Its behavior is then governed by the equations that follow.



Integrator – Fig.2.

The circuit to the left performs the time integration of the input voltage, V_{in}. The values of the resistor and capacitor are scale constants in the resulting output of this circuit. This is the most important analog circuit in this paper – as it makes integration nearly trivial.

$C\frac{dV}{dt} = I_{capacitor}$ Eq.3. I-V Equation – Capacitor (for reference)

The circuit above must maintain the inverting input (the "-" on the operational amplifier) at ground potential, and the only way it can maintain this state is by changing the output of the operational amplifier [1]. This implies the following equations must apply:

$I_{into\ circuit\ (through\ R)} = \frac{V_{in}}{R} = -I_{through\ capacitor}$ Eq.4. Series RC Circuit in Integrator

This equation holds because the resistor and capacitor are in the circuit, and are connected in series. Since the inputs of the operational amplifier do not influence the circuit, the current must be equal through a series circuit. Hence, we may utilize Eq.3. and Eq.4. to find the value of the output voltage, V_{out}.

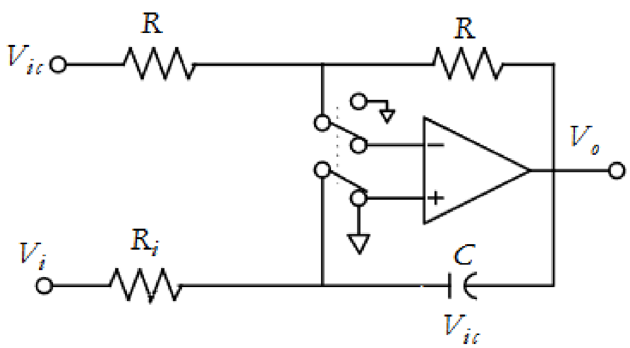
$\frac{V_{in}}{R} = I_{through\ capacitor/resistor} = -C\frac{dV_{out}}{dt}$ Eq.5. Substitution of Eq.3 and Eq.4

From the above equation, we may divide by C and then integrate to find the resulting output of the output voltage for the above system. This leads to the sixth and final equation.

$-\int_{t_{start}}^{t_{stop}} \frac{V_{in}(t)}{RC}\, dt + V_c = V_{out}$ Eq.6. Integrator, equation for V_{out}

The above equation states the output voltage explicitly, namely it is the time integral of the input voltage from the time when the system was set up to the time of observation. In addition, an arbitrary constant - equal to the voltage on the capacitor when the integration began at t_{start}, is added to the output as a "constant of integration". Thus, a simple three element circuit can perform a rather difficult operation [1]. It should be realized that this circuit is very nearly analogous to the electronic equivalent of a bucket that can store water fed through a pipe – as the pressure through a pipe increases, a bucket's water level gradually increases. The "bucket-water" system, in this manner, can form an "integrator" of sorts. However, unlike the bucket, an electronic integrator can integrate very quickly and can be reset nearly instantaneously. It is interesting to note, however, that in the above analogy the constant of integration would be the amount of water that was present in the bucket when the experiment began. Also, "resetting" a "water-bucket" integrator, if such a silly thing were made, would be simply draining its contents. The analogous system in the schematic above can be reset and have its initial conditions applied by means of electromechanical relays, or FET transistors [3].

Although this circuit is not able to provide its initial conditions and reset these conditions, a practical circuit can be designed to do this [3,4]. In practice, this circuit is relatively simple and is shown in figure three below.



Practical Integrator – Fig. 3.

This circuit is a practical integrator, which works on the same principles stated above for the inverting amplifier with a capacitor as its feedback element.

This circuit has two inputs, V_{ic} for the initial condition and V_i for the input voltage. The two switches in the center of the figure are the contacts to a relay system, or are representative or an equivalent FET system which provides the same operation. When the system is in the state in the figure, the noninverting input is at ground potential. Furthermore, the input resistor and the left connection of the capacitor are at ground potential. The inverting input of the operational amplifier, "-" on the triangle above, must then be maintained at the ground potential if possible. Hence, the following equations may apply.

$$\frac{V_{ic}}{R} = -\frac{V_o}{R} \therefore V_{ic} = -V_o \quad \text{Eq.7. KCL applied to the series branch of the circuit above}$$

From this point, it is possible to solve for the output voltage to find that the value of the output voltage is simply the negation of the input voltage, V_{ic}. In this manner, the circuit sets the initial condition across the capacitor, whose other terminal is at ground potential [3]. When the switches in the figure above are thrown into the opposite position, so that they face upwards, the initial condition circuit is grounded and the integrator operates as stated earlier. Thus, a practical integrator is created. Manipulation of the values of the resistors and capacitors can change the rate at which the integration takes place as well [3]. Finally, the value of R in Fig.3. should be smaller than the value of R_i so the capacitor is quickly set with the initial condition – which is the case when the RC time constant is short [1].

From the two preceding circuits, it should be noticed that a "summing integrator" can be designed by combining the two circuits [1,3]. Eq.4. can be rewritten to reflect the summation of the input currents at the inverting input of the amplifier, and the result of the output voltage is the negative time integral of the sum of the input voltages, scaled by the value of $R_{input}C_{amplifier\ feedback}$ for each input [3]. Such a circuit was utilized in this project to allow for fewer amplifiers to be utilized and to minimize the cost of the designed circuit boards.

From this point, it is possible to take sums, differences, products by a constant scalar, and time integrals – all by electronic analog circuits. The next question is likely the obvious "How can these be used to solve useful problems?" or "How does this do any good?". The first question is answered in the next section, and the second is obvious after the first question is answered.

Solving Problems: The "Programming" of an Analog Computer

The circuits above are the "elements" that constitute an analog computer. In fact, they perform mathematical "operations" – and this property gives the "operational amplifier" its name. The operational amplifier networks can be utilized to solve problems in a straightforward manner. The basic procedure is the following process.

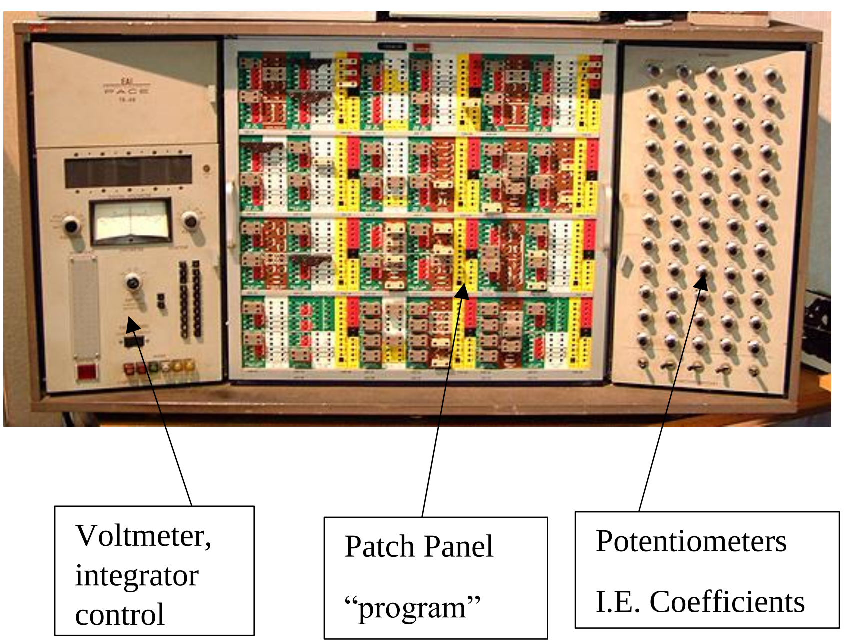
Steps to Using an Analog Computer [3]

1. Identify the equations that must be solved in a problem.
2. Draw a flowchart stating how the elements of the computer must be configured to model the desired problem.
3. Estimate the values of the quantities at hand and scale them if required. If time must be scaled, scale it also.
4. Set up the actual computer to reflect step #3 and step #2.
5. Run the system, recording the results with a sampler or oscilloscope.
6. Analyze the results for accuracy and utility, execute this algorithm again until the desired results are established.

This basic algorithm details essentially all there is to the utilization of an analog computer. Essentially, every problem may be broken down like this. [3] This algorithm is utilized latter in this paper to solve one such example problem from elementary physics. In principle, it should be noted that computing itself is inherently analog – a "digital" computer is very much an analog computer with the caveat that its computational elements operate as saturated or cutoff [4]. In this light, it may be that a digital computer could be utilized that samples the output of an analog computer, sets up the initial conditions, and even perhaps allows for a FPGA like matrix to connect the relevant elements of the machine together. Scaling could also be treated by a digital machine in such a manner. Such a "hybrid computer" is interesting, because if the precision of an analog computer could be made very precise in the future, this may allow for large dynamic systems to be modeled in real time [5]. Furthermore, it should be noted that the human mind itself is a "hybrid computer" in that it contains elements of both digital and analog computation.

Design of an Analog Computer

The introduction and programming sections of this paper detail all the information about this project, except for the *exact* details of how such a machine would *really* be constructed. To this end, some research was conducted on the details of actual analog computers [6]. Such machines utilize the elements presented earlier in this paper, and contained the same basic design throughout various manufacturers [6,7]. Essentially, the machines contained the same set of items: a series of amplifiers and integrators wired to a "patch panel" where the machine can be manually wired to reflect a given problem, input devices – such as potentiometers which store constant values and coefficients, and output devices – such as pen and ink recorders or storage oscilloscopes [3,6,7]. In addition, a digital system was sometimes incorporated to control the setup of the system's initial conditions and the time for which the integrators operated [8]. An image of such a machine is shown in figure four below.



Analog Computer: EAI TR-48

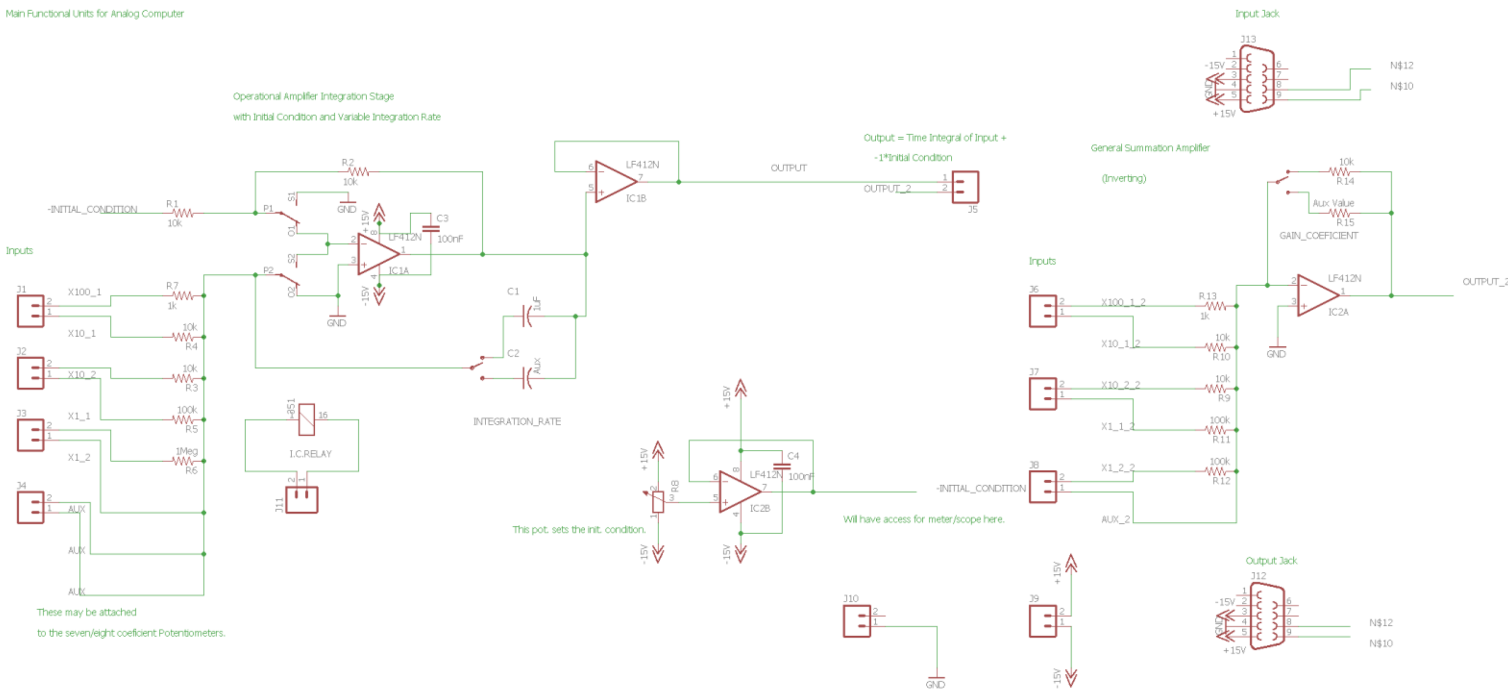
Fig.4.

This machine is a typical analog computer, Ca. 1963-4. The potentiometers to the right provided constants or coefficients, the patch panel in the center was the machine's "program", and the left of the machine included a digital volt meter for numerical output.

Observations of historical analog computers lead to a similar modular design for the printed circuit boards for the analog computer documented in this research paper. From this point, the schematics of the Heathkit EC-1 [9], previous analog computers [10], and intuition from previous electrical engineering courses at The Georgia Institute of Technology were utilized to design the following schematics for the prototype analog computer. Each of the three subunits has many terminals – which acted as the "patch panel" and allowed for the device to be configured differently.

Schematics of the Analog Computer Prototype –

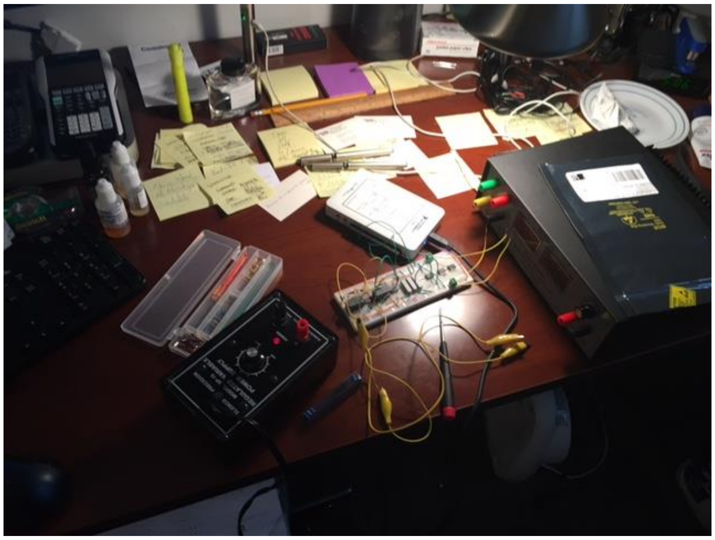
The schematic diagrams were written after a breadboard prototype was tested on several different circuits. The one that functioned superior to the other designs, as determined by a trial and error process, was utilized in the final schematic diagrams. The prototype functioned as one of the three final units would in the final circuit, and a photograph of it is included on the following page.



Prototype Schematics – Fig.5.

The schematic diagram above, made in Eagle 7.2 during November, 2016, was utilized in the system that this paper documents. Several points about the design should be made. First, the terminals allow for quantities to be represented by different factors of ten (for instance, the input x10 to the integrator leads to this variable to be scaled by ten times what the x1 input would yield) – resistors differing by a factor of ten were used to make this possible. In addition, a follower was used to isolate the potentiometer from the low impedance, ten kilo-ohms, of the initial condition circuit. Finally, the D connectors shown in the right of the schematic diagram allowed for the connection of the three boards together without any difficulty. Thus, three integrators and three summation amplifiers are available for use in the prototype analog computer.

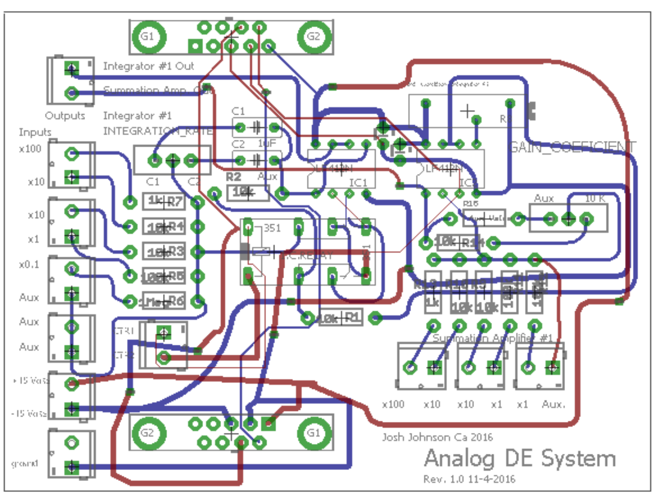
After the schematic diagram was written and tested, the board layout for the system was completed in early-mid November, 2016. This final design was sent to the senior design department of The Georgia Institute of Technology, where the generous assistance of David Steinberg and Kevin Pham allowed for the final boards and parts to be ordered from OSH Park and Digikey, respectively. The items were delivered by December 6^{th}, 2016 and their construction and testing began shortly thereafter – the three boards were completed by December 15th, after final's week for the Fall semester of 2016 was completed and time was available.



Breadboard Test Circuit for Prototype Analog Computer Circuit Boards – Fig.6.

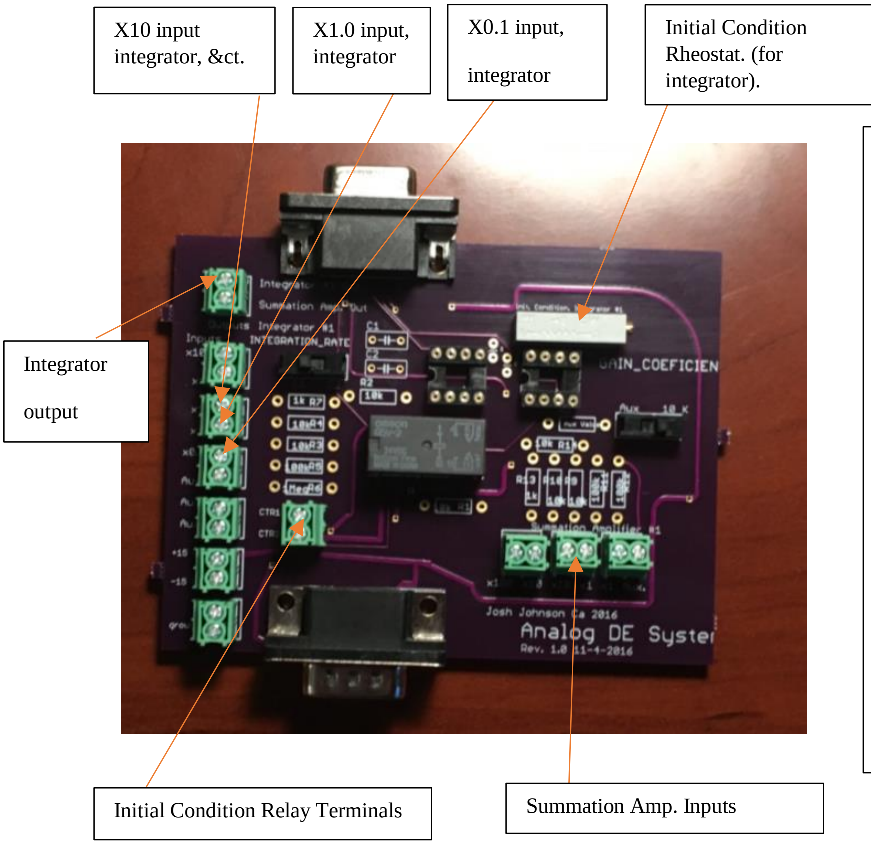
This circuit, photographed on December 3rd, 2016, was the test apparatus that was utilized to perfect the schematic design for the final system. In this photograph, the system is configured to model the trajectory of a body in motion under an initial velocity and the force of gravity. The power supply is towards the right, and the myDaq unit in the center right read off the output voltage values.

For the sake of precision, the final board layouts for the printed circuit boards are copied here. Gerber files were exported from these designs and sent to OSH park for fabrication. This service did their job adequately, allowing for this prototype analog computer system to be completed before the end of the Fall Semester of 2016. Notice how the power traces and relay traces between the boards are thin – only 10 mils or so. This was a flaw on the part of the author. The autorouter was employed, and it did a terrible job. This was done due to a lack of time, considering my current studies. However, the boards function as desired despite this.



Printed Circuit Board Plans – Fig.7.

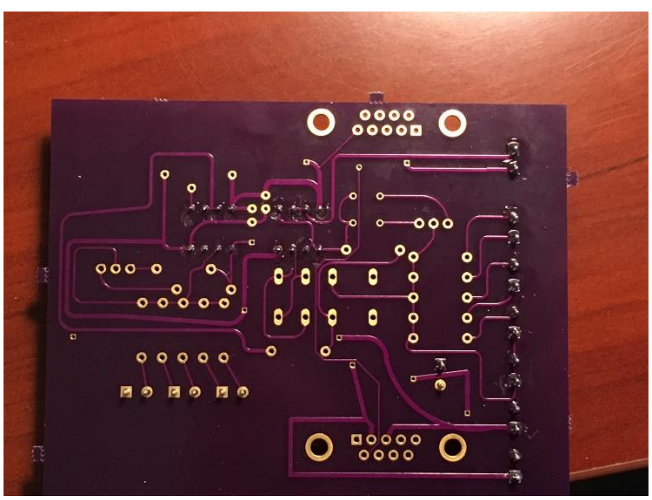
The photograph to the left shows the printed circuit board layout designed for the analog computer prototype. The red layer is on the top of the printed circuit board, where the components are placed and soldered, while the blue layer is on the bottom of the board. The design is inelegant and hasty, but gets the job done. The square blocks with the two lines on the right of their symbol are the terminal blocks that the machine utilizes to connect the operational amplifier circuits together. Notice that the system allows for a single pole double throw relay switch to alter the feedback elements of the circuit – allowing for different integration rates and gain constants for the integrator/summation amplifier, respectively.



Final Assembled Circuit Board – Fig.8.

This photograph shows the first of three printed circuit boards constructed for the analog computer prototype. This unit was the first one completed, and all of them were assembled by December 15th, 2016. Shortly thereafter, the system was utilized to model the trajectory of a bouncing ball. However, I did not have a storage oscilloscope – so the results of calculations on the machine could not be appreciated until latter. Finally, notice how the operational amplifiers are not inserted into their pin sockets yet in this image. Such sockets were added so no soldering would be required if one broke.

During the final days of the fall Semester of The Georgia Institute of Technology, the system was assembled and tested. The final system, with some relevant remarks, is included below. The device worked as specified, after a few soldering errors were discovered in one of the first boards.

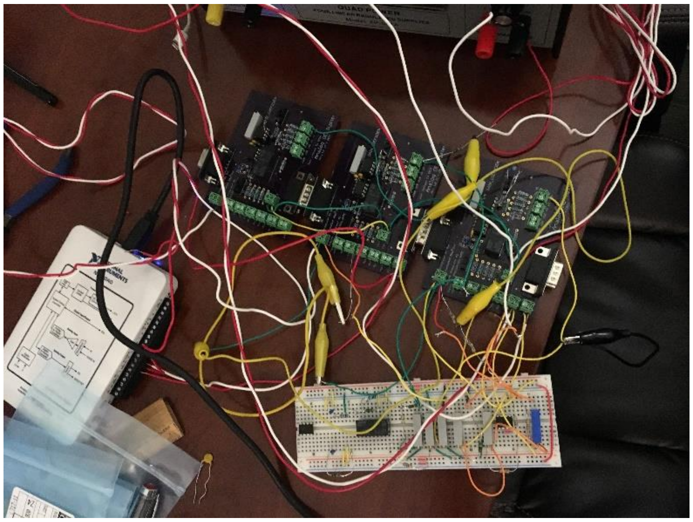


Printed Circuit Board During Assembly

- ***Fig. 9.***

This image shows one of the three printed circuit boards during assembly – where through-hole components are inserted into the board and soldered to the back of it. Notice how the back of this board’s traces coincide with the blue traces on the earlier layout design shown in Fig. 7.

From this point, the final system was completed. Soon thereafter, the system was programed to solve the equations of a bouncing ball – although this is not documented in this paper. The completed system is shown in the following image – it was ready for its first “program”. One such problem is detailed in the following section.



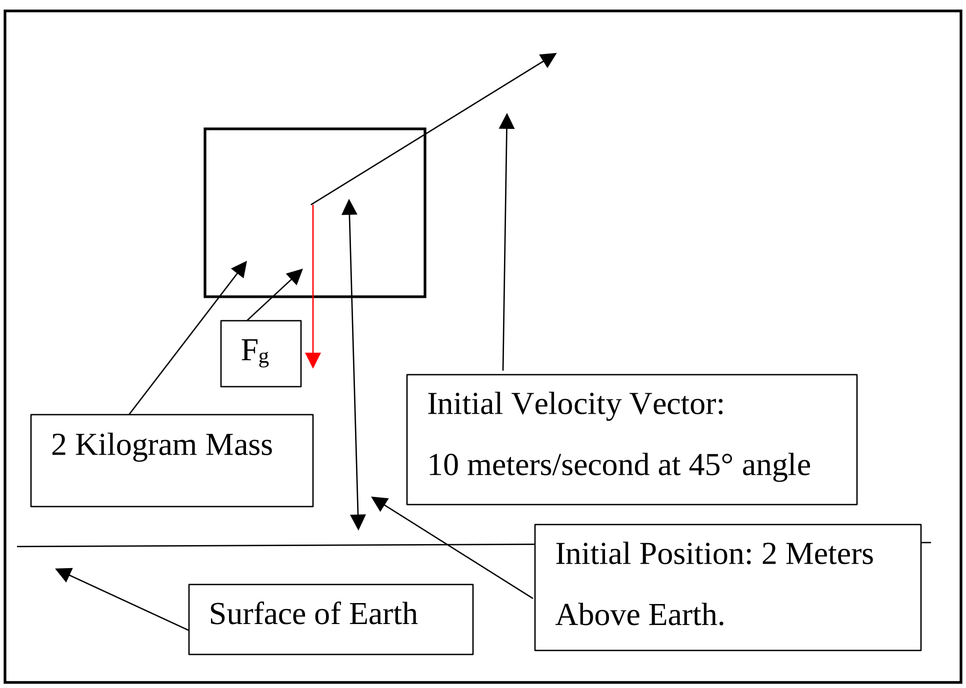
Final Analog Computer Prototype wired to model bouncing ball system – Fig.10.

This image shows the completed analog computer prototype. Notice that it consists of three of the boards as detailed in the schematic diagrams and board layouts above. This particular system, hastily constructed due to time constraints, modeled the trajectory of a bouncing ball – it actually utilized the exact same system that was documented in the Heathkit EC-1 manual [7,9].

As the above images detail, the system was completed during mid-December 2016 and it was possible to solve differential equations with the system, as detailed in the procedure above [3,9]. Thus, this paper studies one such stereotypical system, the falling body problem from introductory physics.

The Falling Body Problem solved on an Analog Computer – First Experiment

A body with an initial velocity under the influence of gravity is a common problem solved in elementary physics. Such a system is given below, with its subsequent free body diagram and equations. Notice that the problem assumes that the body is under the influence of a constant acceleration due to gravity, meaning it is near the surface of the earth [10].



Falling Body Problem – Fig.11.

The two-kilogram mass in this free body diagram initially has a velocity of 10 meters/second at a 45-degree angle with the horizon. It starts at a position that is two meters above the ground. The only force acting on the object is the acceleration due to gravity. The question that must be solved is: "What is the trajectory of this object given these initial conditions?". Such a problem is easily solved by the prototype analog computer designed in this paper.

The problem may be solved as in physics I, where Newtonian mechanics was introduced. First, the initial velocity must be divided into its x-axis and y-axis components. This is done as follows:

$V_{initial-x\ component} = 10(\frac{m}{s})\cos 45° = 7.071\ meters/second$ Eq.8. Initial Velocity – x-axis

$V_{initial-y\ component} = 10\left(\frac{m}{s}\right)\sin 45° = 7.071\ meters/second$ Eq.9. Initial Velocity – y-axis

Since these components of the initial velocity are known, it is possible to utilize Newton's Second Law and known kinematic equations to model the trajectory of the object as it moves through space [9]. Thus, the following equations may be stated.

$F = MA = M * 9.8\frac{meters}{second} \rightarrow For\ an\ object\ near\ the\ surface\ of\ earth.$ Eq.10. 2nd Law

It is known that the acceleration due to gravity is nearly 9.8 meters per second, constant, on the surface of earth. Since this always points toward the ground, it acts in the y-axis direction. There is no acceleration in the x-axis direction, at least for this problem where wind resistance and friction are neglected. From this point, we realize that acceleration is the derivative of velocity and velocity is the derivative of position. Thus, it is true to state the equations below.

$\frac{dp}{dt} = v\ ,\ \frac{dv}{dt} = \frac{d^2p}{dt^2} = a$ Eq.11.1-11.2 Velocity is the derivative of position, &ct.

Notice in the above equations, p is the position of an object, v is its velocity, and a is its acceleration. From these equations, it seems that if acceleration were integrated *twice*, then the position of an object would be known. Of course, the initial conditions of the problem at hand would have to be known for this to work correctly. However, we may state Eq. 11. as stated earlier below.

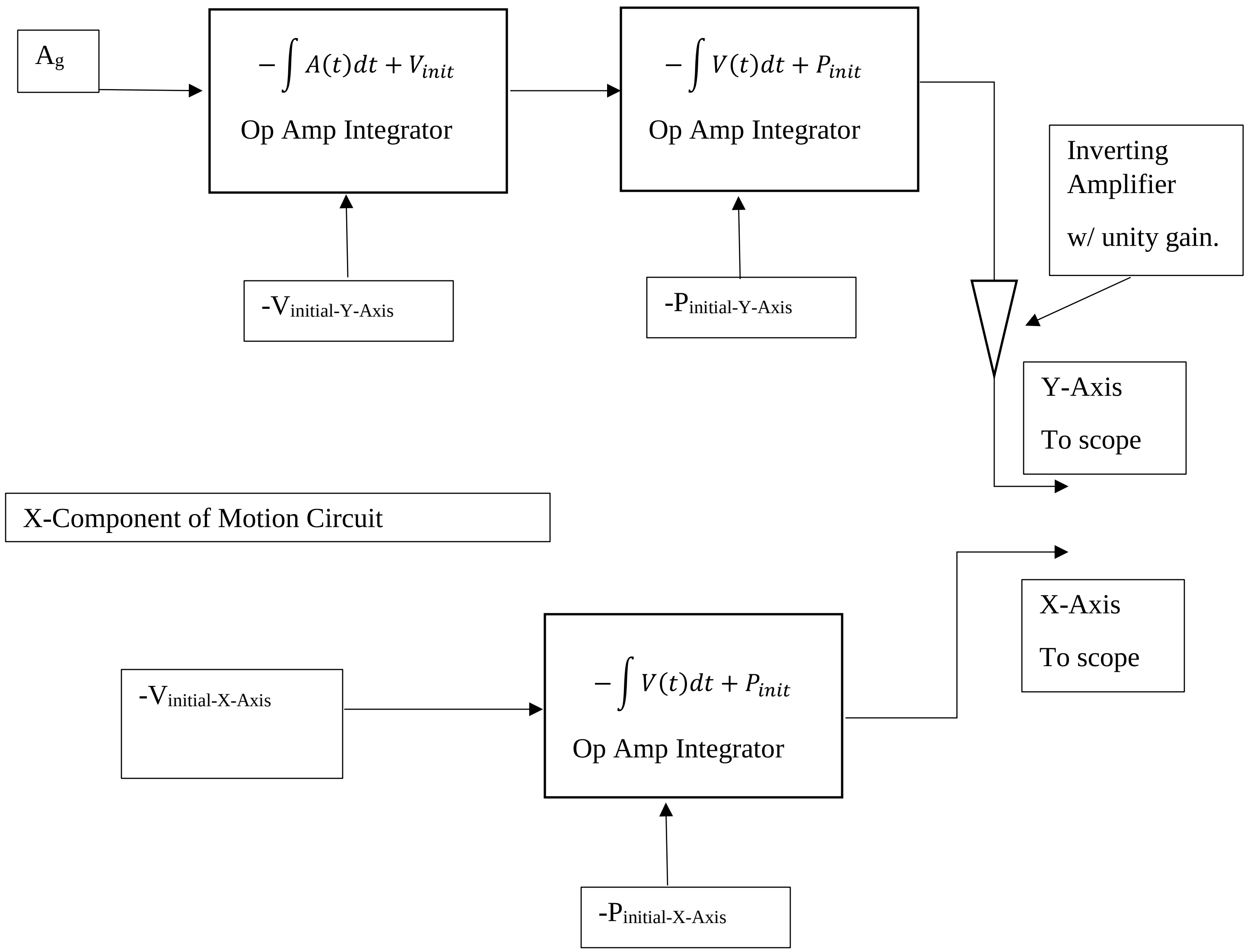
$\int acceleration(t)dt + A = velocity$ Eq.12.1 – Finding velocity from acceleration.

$\int velocity(t)dt + B = position$ Eq.12.2 – Finding position from velocity

Notice in the above equations, A and B are the initial velocity and the initial position respectively. For the analog computer utilized in this project, these are the initial conditions applied to the operational amplifier integrators. From these observations, although we do not know the *analytic* solution to the problem, we have stated all that is required to set up the analog computer to solve the problem. From this point, we may draw a flow chart that reflects the general idea written above: integrate acceleration (if nonzero) and add the initial velocity as its initial condition, then integrate this value to find position. Notice that the initial condition of the second integrator is the initial position of the object. This process may be done for the x-axis and y-axis components, and with a storage oscilloscope the trajectory of the object may be found. The next page shows the flow chart that details this idea schematically – the analog computer in this project was set up based on this diagram and the solutions were recorded from this.

Flow Chart of Analog Computer for Falling Body Problem –

Y-Component of Motion Circuit



Flow Chart of Falling Body Problem on Analog Computer – Fig.12.

For the topmost circuit, the acceleration due to gravity is integrated to give the negative of the velocity in the y-axis direction. This is fed through a second integrator to find the position of the object in the y-axis direction. The inverting amplifier is added to yield a sign change. Below this circuit is the integrator that integrates the constant velocity in the x-axis direction to yield the x-axis position of the projectile. Thus, the system models the problem of the falling object introduced earlier.

The flow chart above is the "program" for an analog computer. Note that each of the boxes indicates an operation, constant, or another function that can be executed by the circuit elements given earlier in this paper. From this work, it is possible to wire the analog computer to implement the problem at hand, in effect to become an "analog" of the problem – with voltage analogous to the quantities in the actual problem. This is where the methodology of "analog computing" gets its name, not from the use of analog electronics – but since the system models a problem by this "analogous" construct.

A Note on Accuracy -

Before the program was set up and run, an analytic solution to the introduced falling body problem is needed so that the accuracy of the analog computer can be determined relative to the known solution. 1% tolerance components were used in the machine, and precision low drift and high bandwidth operational amplifiers (LF 412's) were utilized to minimize unideal analog effects that the system may encounter. Furthermore, the voltage values in the system were checked with a digital volt meter to ensure they are set to the precise values that are required. Despite this, however, the system did have an issue with high-frequency noise and other interference. A low-pass RC filter was added to the output of the final stages to prevent this noise from damaging the output values as recorded by the Rigol 1102 storage oscilloscope. It appears to yield results within a reasonable margin of precision and error – likely limited by the quality of the components in the machine's construction.

Analytic Solution to Falling Body Problem –

For certain problems, an analytic solution is difficult to impossible to find [11]. In such cases, approximation by a digital computer program or an analog computer is the only means to analyze such systems – which may be chaotic in nature. However, the falling body problem introduced here is not such a system – it is possible to find an analytic solution for the problem. This is found by direct integration as follows below.

$\int -9.8dt + V_{init-Y-Axis} = Velocity = [-9.8t + 7.071]m/s$ Eq. 13. Velocity – Y-Axis

When this function is integrated again, position along the y-axis may be found. That is:

$\int -9.8t + 7.071dt \ + \ P_{init-Y-Axis} = -4.9t^2 + 7.071t + P_{init-Y-Axis}$ Eq. 14. Position, Y-Axis

From the initial condition that the object begins at 2 meters above the ground, it can be shown that the value of the y-axis component of the position of the object is the equation stated as follows.

$Position_{y-axis} = \ -4.9t^2 + 7.071t + 2 \ \ meters$ Eq. 15. Position of the projectile, Y-Axis

This same idea may be used to solve for the x-axis position value as a function of time. Proceeding as before, only a single integration is required since the x-component of acceleration is zero. Thus, a single integration of velocity may yield the x-component of the projectile.

Analytic Solution of Falling Body Problem (continued)

$\int 7.071dt + P_{init-X-axis} = Position_{x-axis} = 7.071t + 0$ Eq. 16. Position X-Axis.

Scaling the Values for the Analog Computer

From this manipulation, an analytic solution is known for the system. Now the first two steps of the earlier flow chart have been executed. Now the values for the analog computer must be scaled if they do not fit into the range available to the machine, which is fixed by the power supply voltages of +/- 12 Volts. The range of the values involved in the problem may be found using established equations for projectile motion [12]. These are stated below.

$Height_{max} = \frac{(V_{magnitude}Sin(\theta))^2}{2*g} = \frac{(5*\sqrt{2})^2}{19.6} = 2.55102\ Meters$ Eq.17. Max Height Projectile.

Notice in the equation above, θ is the angle at which the initial velocity vector is positioned, in this instance 45 degrees. Also, $M_{agnitude}$ is the magnitude of the velocity vector, or 10 m/s. The rest of the variables follow from the earlier equations, for instance g is the acceleration due to gravity.

Also of interest is the range at which the projectile travels, which sets the scale factors (if they are required) on the x-axis quantities in the computer. The equation for the range that the projectile travels is well known as the following equation [12].

$Range = Max - X - Axis\ Distance = \frac{V_{magnetude}{}^2}{g}\sin(2\theta) = \frac{100}{9.8} = 10.204\ meters$ Eq.18.

The values are reasonable if a 1 volt = 1 meter scale factor is introduced. This is very easy for this problem, a rather simple problem. However, this is always not the case and sometimes several scale factors must be introduced. The process is the same, however. If the maximum values in a problem are not known, then they can be estimated for this purpose [3]. If they are not correct, and the output is too small to be reasonable or causes the amplifiers to saturate, then the process can be repeated in a trial-and-error type of fashion.

Since the magnitude scale factor has been set (1volt = 1 meter for both x and y axis values), a time scale value must be set for the system as well. One reasonable choice is to set the system to run in “real time” where one second in the problem is also a single second in the computer’s time. This need not be the case, however, and altering the value of the capacitors in the machine’s integrators can scale time like other quantities in the machine. The system is run in real time by using a 1μF capacitor as the feedback on the integrator and a 1Meg ohm resistor as the inputs to the integrator – this leads to the time integration of the voltage with respect to time in that the integrator increases its output value by one volt every second that a constant one volt input is applied. Now that the scale quantities are set, all that is left is to run the system on the machine and test the results that are ascertained. This is detailed in the next section.

Programming the Analog Computer

The machine must be wired to reflect the falling body problem system. This is accomplished by manually wiring it to reflect the block diagram given on page 12. The x0.1 input of the y-axis velocity integrator is connected to a potentiometer that has +9.8 volts across it. The initial condition of this integrator is -7.07 volts, of the initial y-axis velocity. The output of the y-axis velocity integrator is then fed into the x0.1 input of a following integration stage, see the figure on page nine to see where these input terminals are located on the machine. The position integrator on the y-axis system is then set to have an initial condition of +2 volts, or the initial condition for the y-axis position. This completes the y-axis circuit and the output is connected to the y-channel input of the x-y storage oscilloscope, after it is passed through an inverting amplifier (made with a summation amplifier, with only one input) and a low pass filter made with a 10k Ohm resistor and a 0.1uF Capacitor. The output thus has the correct polarity and is free from high frequency noise.

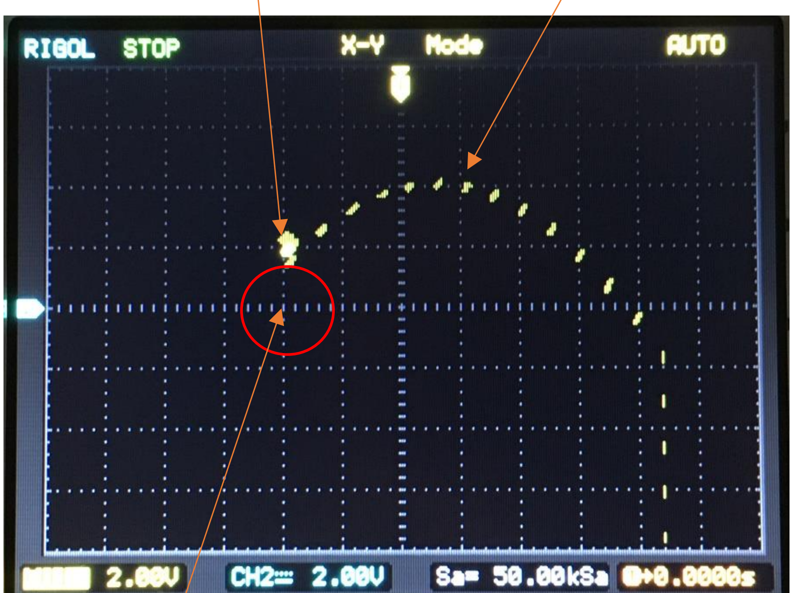
The x-axis system is wired in a similar fashion, the single integrator has a x0.1 input (which uses a 1Meg Ohm resistor for the resistance connected to the input terminal of the operational amplifier – see the schematic diagram given earlier) attached to a potentiometer which feeds -7.07 volts into this input. The initial condition of this integrator is set to zero volts, and the output is passed through an identical low-pass filter as detailed for the y-axis system. Now the system is completed and ready to solve the established falling body problem. The output plots are attached in the figure below, acquired when the system was run.

Results

Initial Position

X = 0, Y = 2 volts.

The Body's Trajectory



The Origin – Shifted here.

Output of Prototype Analog Computer – Fig.13.

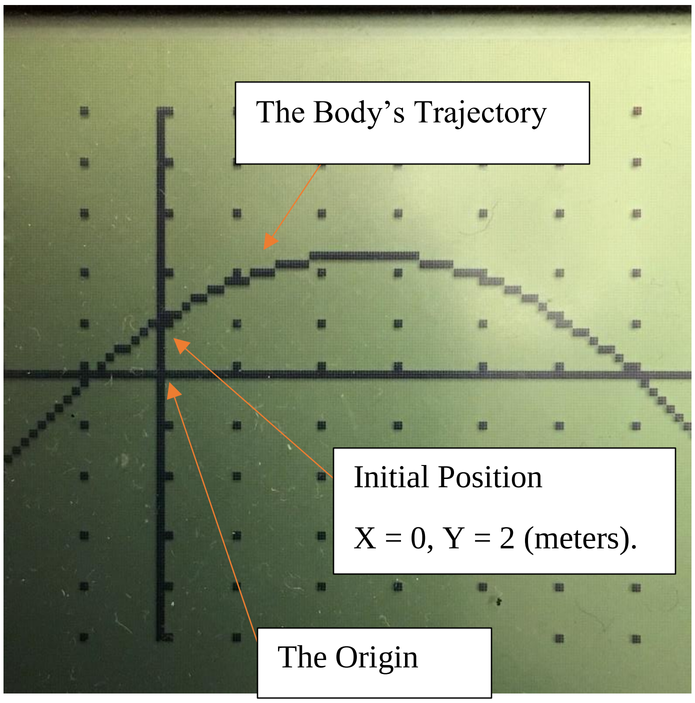
This is the output of the analog computer, as recorded by a Rigol 1102 storage oscilloscope. The scale on this plot is two volts per division for both the x and y axes. Notice that the object is shown as moving in a parabolic trajectory that has a peak of about 2.25 volts and a range of 10 volts. Since 1 volt = 1 meter in this system, these agree with our expectations.

Scale Factors for Analog Computer:

1 Volt = 1 Meter, x and y axis. Plot: 2 volts/div.

The system was solved by the prototype analog computer, which answers the first question of this project and part of the remaining questions. It is possible to model a dynamic system with analog electronic circuits. The accuracy is imperfect, albeit decent. Notice how there is a line on the right of the figure above. This is not an error on the computer's part – instead the x-axis integrator is saturating at this point and the system is unable to represent a further x-position. This could be remedied by using higher voltages in the system, or by scaling the problem as detailed earlier. Notice the projectile starts at one unit above the origin, or two volts. This is the initial condition given earlier.

Comparison with the results with the actual solution



Solution to Falling Body Problem – Actual solution as found by TI-84 – Fig. 14.

Notice how this plot shows the same characteristics as the plot given by the analog computer above. The scale is identical, at two volts per division. It appears that the analog computer computed a very similar trajectory to what this digital device could do – with an error of 8% or less (see error analysis section, below). This experiment makes an analog computer seem like an analog graphing calculator, or electronic slide rule, which essentially captures the central idea of what the thing does. This plot was added here for comparison with the results as posted in Fig. 13.

Error Analysis – First Experiment

Several characteristics are measured off both plots for comparison. From these values, the percent error of the analog system can be found. This should, theoretically, be within 1% or so – but could be better or worse in practice. Thus, the measurements and calculations are stated.

Error Analysis - Analog Computer vs. Actual Solution - Table.1.

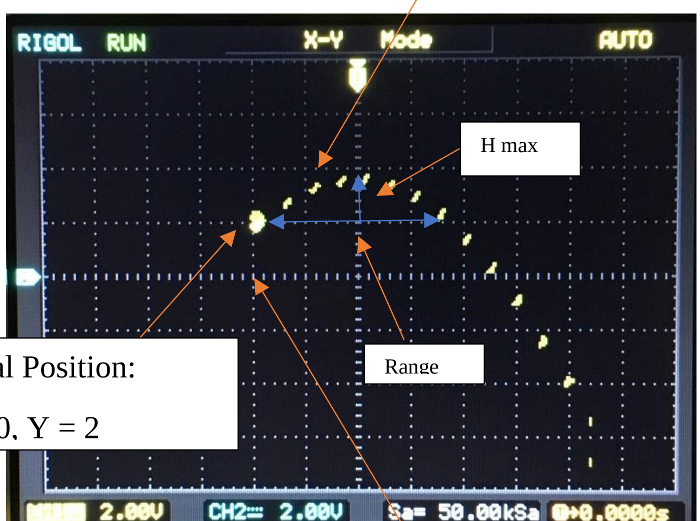
Measurement	Analog Plot	Actual Solution	Percent Error (%)	Units - Measured Quantity
Peak Value – trajectory (H_{max})	2.3	2.55	9.803921569	Meters
Range - X-Axis	9.9	10.2	2.941176471	Meters
Run Time - Actual System	1.7	1.44	18.05555556	Seconds

The percentage error is within 10%, at least for the measurements of the distance quantities. The run-time is off by the largest margin, 18%. This might be due to the stopwatch operation that was utilized to make this measurement. The reason for the discrepancy in the dimension values may indeed be described by a voltage source that was later found to vary by 10% in its operation. The average percent error of the system is 10.26%, roughly equivalent to that of a slide-rule calculation. However, an analog computer can model dynamic systems in real time while a slide rule is limited to static calculations.

Before any major conclusions are drawn regarding the viability of analog computing, another trial run was tested on the system. This serves two purposes, to determine if the above results are indicative of the system's behavior, and to show the use of an analog computer in a problem where the parameters are varied to see the results of a certain theoretical situation.

Second Falling Body Problem – g is changed to 12.0 meters/second^2

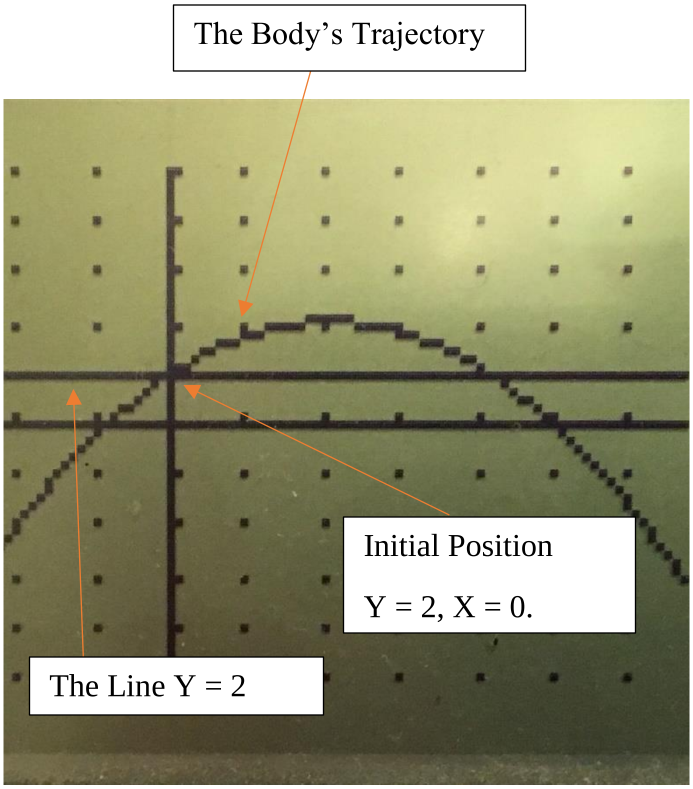
For the sake of brevity, the exact same problem parameters were kept, except for the alteration of g to a value of 12.0 m/s^2. The system was wired in the exact same manner, because the equations were the exact same system. After the computer was run, the plot was taken and is reprinted in the figure below.



Falling Body Problem II – Fig.15.

This system is the exact same falling body problem, with the corollary that the acceleration due to gravity is fixed at 12.0 m/s^2, as opposed to its previous value of 9.8 m/s^2. This would be the case if a projectile were thrown on another planet, with a different mass and geometry than earth. Although such a system may not be seen by terrestrial human beings, the machine can still model it readily.

The actual solution for this can be found by interchanging the constant of acceleration in the earlier equations from 9.8 to 12.0 meters/second^2. When these manipulations are made, a graphing calculator can compute the resulting trajectory. This was accomplished for this problem and the resulting plot is on the following page.



Actual Solution – Falling Body Problem II – Fig.16.

This is the actual solution to the second falling body problem – added to compare the accuracy of the analog computer to a digital computer. Notice that the graphs again show a similar parabolic trajectory. They are quantitatively similar – as is shown in the following table for the error analysis. The line in the center of the image is the line y = 2. Notice that the projectile falls faster when the force of gravity is increased, as can be expected in this situation – which would mean that the planet the projectile is located on would have more mass than earth.

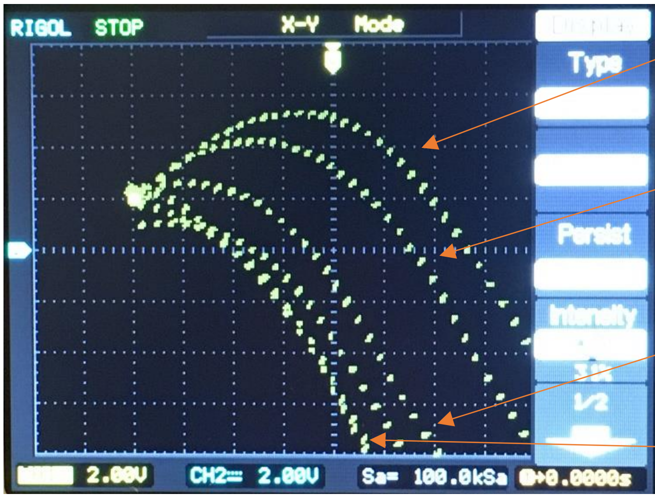
Error Analysis – Falling Body Problem II

Error Analysis - Analog Computer vs. Actual Solution - Table.1.				
Measurement	Analog Plot	Actual Solution	Percent Error (%)	Units - Measured Quantity
Peak Value – trajectory (H_{max})	1.9	2.08	8.6538	Meters
Range - X-Axis	7.60	8.33	8.7635	Meters

For this instance, the run-time value was omitted as it is believed that human error and oscilloscope sampling delay contributed to the discrepancy of this quantity. Notice how the error values from this experiment are still close to 10%. It appears that a different design, better workmanship, or more precision components would be required to construct a better machine. Notice that although 1% resistors were utilized in this machine, since dozens of them were linked together to build the machine it is possible that the error accumulated – resulting in the 10% discrepancy reported in the table above.

An analog computer can be useful in its ability to simulate multiple solution trajectories for a certain problem in very little time. For the above falling body problem, this is demonstrated in the following figure. In this instance, the initial velocity of the body is altered while gravity is left at a constant value, g. The different solution trajectories are shown, and their initial conditions are related next to the figure. Such solution sets can be computed instantaneously if a digital computer were utilized to provide the control signals for the machine, initial conditions, and to time the operation of the integrators in the machine [8,10].

Third Experiment – Falling body problem solution set for different initial conditions



Soln. #1. V_x = 7.07 m/s, V_y = 10 m/s.

Soln. #2. V_x = 7.07 m/s, V_y = 7.07 m/s.

Soln. #3. V_x = 7.07 m/s, V_y = 3.00 m/s.

Soln. #4. V_x = 7.07 m/s, V_y = 0.00 m/s.

Four Solution Curves for Falling Body Problem – Fig.17

The analog computer can compute many solution trajectories for the falling body problem in response to different initial conditions. The y-component of the initial velocity is altered above to yield a family of such curves. A digital computer can operate the analog computer so that these solutions can be viewed at once on an oscilloscope/sampler. Notice the acceleration due to gravity is fixed at 1g (9.8 meters/second = 9.8 volts) and the initial condition is y = 2, x = 0 (the above graphs are shifted) for all solution trajectories.

In addition to being useful as a model of physical systems, analog electronic circuits can be utilized as a useful classroom tool and apparatus for demonstration purposes. One such application is detailed in the following section.

The use of Analog Computing with a Chaotic System of Equations

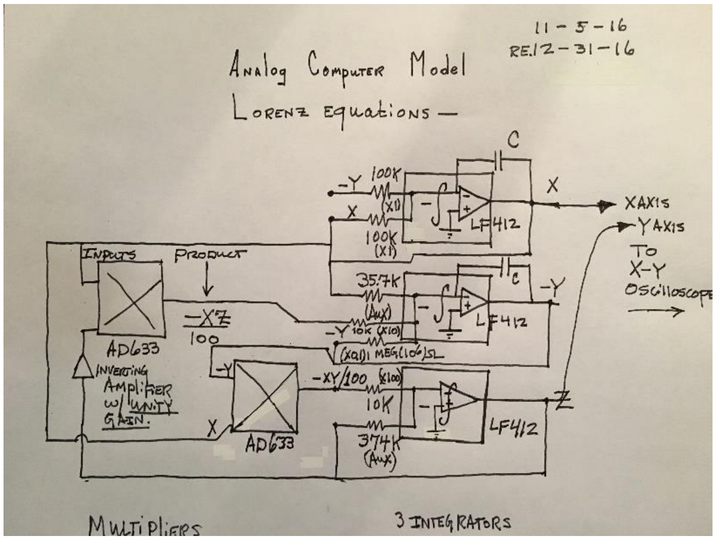
The Lorenz differential equations, a set of three linked differential equations, has a chaotic solution trajectory under certain initial conditions and input parameters [11]. This system of equations has the following form:

$\frac{dx}{dt} = 10(-x + y)$ Eq. 19.1. Lorenz Equations First Equation.

$\frac{dy}{dt} = 28x - y - xz$ Eq. 19.2. Lorenz Equations Second Equation.

$\frac{dz}{dt} = -\frac{8}{3}z + xy$ Eq. 19.3. Lorenz Equations Third Equation.

These equations form a nonlinear, autonomous three-dimensional system [11]. Such a system can be demonstrated with the utilization of analog circuits to model the system above.



Schematic of Lorenz Equation Analog Circuit Model – Fig. 18.

This image shows the schematic of the Lorenz equation circuit model, which calculates the x,-y, and z values of the solutions to the above set of equations. The terms are summed and then integrated by the three summing integrators. The multipliers on the left-hand side of the figure provide the nonlinear terms that are required by the system. The output can then be viewed in real time on an oscilloscope.

The above circuit contains resistors that act as the coefficients in the Lorenz equations. The first integrator contains two input resistors of 100k ohms each, which have their inputs from the x and -y inputs. These are summed, negated, and then integrated. The result is ∫10(-x+y)dt = x. The same process is executed by the remainder of the circuit. Since each of the integrators has an output that is proportional to the reciprocal of $R_{input}C_{feedback\ capacitor}$, the values of the resistances in the circuit are altered to reflect the relative magnitudes in the above differential equations. The equation that governs this “scaling” is below.

$\frac{1}{10^{-6}R} = coeficient\ in\ above\ equation$ Eq. 20. Lorenz Equation Scaled Resistance Values.

This equation leads to the values of 100k ohms in the first circuit above. Furthermore, it can be utilized to determine that the y-integrator should have resistance values on its input of 35.7k ohms, 1 Meg ohm, and 10k ohms for the x, -y, and product term -xz/100 respectively. Furthermore, the values of the input resistances in the final integrator for the z variable are 10k and 374k Ohms by the same principle.

Since this system is designed, it can be constructed to observe typical solution trajectories for the differential equation. This was accomplished in November, 2016 for Dr. Chen’s 2552 Introduction to Differential Equations Course at The Georgia Institute of Technology. Dr. Chen was very generous with his time and assistance, and it was a true honor to be able to show this experiment to his students when the “chaos” section was studied latter in the fall, 2016 semester.

From this application, it is apparent that analog computers can be utilized in a variety of settings and contexts. In addition, it is obvious that they can be utilized for demonstration purposes. Their main limitations include difficulty of programming and limited accuracy. After these two concerns, a limited range of quantities is also a difficulty to circumvent. However, if these three problems could be rectified, then analog electronic circuits could be utilized for a variety of tasks and purposes [10].

Conclusions

Although analog computation is a viable means to simulate dynamic systems, differential equations, and other operations, it remains plagued with difficulties. Despite much effort in this area, the percent error for such machines remains at 10% or so. Perhaps this can be reduced, maybe in the future. If this were to occur, then there may be specialized applications for such devices – as in instances where power is at a minimum or where simplicity is required. Until such time, this area represents a dead end as it is impossible to proceed without great difficulty. However, this research was not in vain – as it shows that alternate computational paradigms are possible with different systems of representation, operations on information, and input/output devices. Furthermore, it was realized that analog computation can model the qualitative information of a system within 10% of the correct values. Thus, for applications where this error is negligible or can be tolerated, such devices can be utilized. In addition, analog computers can show qualitative behavior of differential equations very close to the expected curves as graphed by a digital computer. If it were possible to eliminate the "patch panel" nature of programing an analog computer, and to use a digital system to minimize the machine's error, then it would be possible to utilize analog computation as a high-speed and concise engineering tool.

In conclusion, this project answered the questions asked in the introductory section. It is possible, with some moderate difficulty, to design an analog computer and to solve physical problems on it. It can solve these problems, with an error of about 8%-10% under normal circumstances. If a large scale analog computer were constructed, great care would have to be taken in regards to component precision, analog noise, and other unideal aspects, to make such a system viable and able to perform better than a digital computer. Now in my studies, such a task is beyond the realm of my abilities. However, this is a perfect springboard for such a future project. Perhaps it is possible that a carefully designed analog computer may have some use in the future of research and development, industry, or consumer electronics?

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