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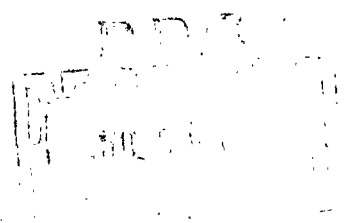
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**PROGRAMMING AN ANALOG COMPUTER  
FOR A LARGE CLASS OF TRAJECTORIES**

Albert I. Talkin

20 June 1963



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DA-5N03-01-003  
AMCMS Code 5010.11.71200  
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A LARGE CLASS OF TRAJECTORIES

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FOR THE COMMANDER:  
Approved by

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## ABSTRACT

The negative gradient method is extended to the stable analog computer programming of a class of time varying trajectory problems defined by  $\phi(x,t) = 0$ ,  $f(x,u,t) = 0$ ,  $u = \dot{x}$  where  $x$  and  $u$  are  $n$ -dimensional vectors,  $\phi$  is an  $m$ -dimensional position vector function ( $m < n$ ), and  $f$  is an  $(n - m)$  dimensional velocity vector function. This class of problems includes the amplitude-stabilized oscillator, two-dimensional contour tracing (ref 1), conic section generation and three-dimensional trajectory plotting when at least the first integral of the equations of motion are available.

An augmented velocity vector function is defined

$$f' \equiv f + \frac{d\phi}{dt}$$

which provides  $n$  independent equations  $f' = 0$ . For a fixed  $x$ , these equations can be solved for  $u$  in the computer reset mode. In the compute mode  $u$  is connected to the integrator developing  $x$ . The resulting system is analyzed and shown to be easily stabilized.

## 1. INTRODUCTION

### 1.1 Definition of the Problem

The problem will be defined in  $n$ -dimensional space. Vector/matrix notation will be employed to reduce the labor of writing equations. The following definitions will be used:

- (1) position vector  $x \equiv [x_1, x_2, \dots, x_n]$
- (2)  $i$ -th position function  $\phi_i \equiv \phi_i(x,t)$ ;  $i = 1 \dots m$ ,  $m < n$   
( $t$  is the independent variable time)
- (3) position function vector  $\phi \equiv [\phi_1, \phi_2, \dots, \phi_m]$
- (4) velocity vector  $u \equiv [u_1, u_2, \dots, u_n]$
- (5)  $j$ -th velocity function  $f_j \equiv f_j(u,x,t)$ ;  $j = m + 1, m + 2, \dots, n$
- (6) velocity function vector  $f \equiv [f_{m+1}, f_{m+2}, \dots, f_n]$
- (7) position error vector  $v \equiv [v_1, v_2, \dots, v_n]$
- (8) velocity error vector  $a \equiv [a_1, a_2, \dots, a_n]$
- (9) gradient vector  $\nabla_x \equiv \left[ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right]$
- (10) gradient vector  $\nabla_u \equiv \left[ \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots, \frac{\partial}{\partial u_n} \right]$
- (11) general matrix notation:  $A \equiv (a_{ij})$  and  $I \equiv$  unity matrix  $\equiv (\delta_{ij})$
- (12) diagonal gain matrix  $K \equiv (k_i \delta_{ij})$   $i, j = 1 \dots n$
- (13) diagonal gain matrix  $G \equiv (g_i \delta_{ij})$   $i, j = 1 \dots n$

(14) scalar or inner product of two n-dimensional vectors (p,q):

$$(p,q) \equiv p_1 q_1 + p_2 q_2 + \dots + p_n q_n$$

(15) positive definite matrix  $\equiv$  a real symmetric matrix all of whose characteristic roots are positive

Using the above definitions the problem can be stated as follows: Given the set of independent equations

$$\phi(x,t) = 0 \quad (1.1)$$

$$f(x,u,t) = 0 \quad (1.2) \quad (1)$$

$$u = \dot{x} \equiv \frac{dx}{dt} \quad (1.3)$$

program an analog computer to generate the unknowns  $x(t)$ ,  $u(t)$ . (1.1) represents m equations, (1.2) represents n-m equations and (1.3) represents n equations for a total of 2n equations in 2n unknowns  $x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_n$ . The problem has been formulated in terms of  $u$  and  $\dot{x}$  rather than  $\ddot{x}$  and  $x$  because of the distinction between the computer variable,  $\dot{x}$ , which by definition is the total input to the  $x$  integrator and the problem variable  $u$  (i.e. the  $\dot{x}$  required as a solution to the mathematical problem.). A full statement of the problem also requires that n-m of the position coordinates be specified as initial conditions. Before continuing with the analysis of such a system, it may be helpful to expand upon the terse problem statement by considering some examples and applications in two- and three-dimensional space.

## 1.2 Examples and Applications

In this section vector notation is momentarily abandoned in favor of more conventional  $x,y,z$  notation. Also  $u$  will be eliminated from the equations by the substitution  $\dot{x} = u$ .

Consider the three-dimensional system (2):

$$\phi_1(x,y,z) \equiv x^2 + y^2 - c^2 z^2 = 0$$

$$\phi_2(x,y,z) \equiv \alpha x + \beta y + \gamma z + h = 0 \quad (2)$$

$$f_3(\dot{x}, \dot{y}, \dot{z}, x, y, z) \equiv \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - s^2 = 0$$

(c,  $\alpha$ ,  $\beta$ ,  $\gamma$ , h, s are arbitrary constants)

The simultaneous solution of (2) is a conic section traced at constant speed  $s$ . In (2) the variable  $t$  does not appear explicitly in  $\phi$  or  $f$ .

Consider the two-dimensional system (3):

$$\phi_1(x,y,t) \equiv x^2 + y^2 - r^2(t) = 0 \quad (3)$$

$$f_2(\dot{x},\dot{y},x,y,t) \equiv \dot{x}^2 + \dot{y}^2 - s^2(t) = 0$$

If  $r(t)$  is constant and  $s(t)$  is a linear ramp, (3) represents linear frequency modulation without any amplitude modulation. If both  $r(t)$  and  $s(t)$  are constant, (3) represents an oscillator with highly stable amplitude and frequency characteristics. If  $r(t)$  is constant and  $s(t) = \frac{d\theta}{dt}$ , (3) performs trigonometric resolution (ref 2). If  $s(t)$  is a constant, (3) represents a complex modulation scheme obeying the law (AM) x (FM) = constant. The modulating intelligence  $r(t)$  can be received by either an AM or FM receiver and is redundant.

Consider the system (4):

$$\phi_1(x,y,t) \equiv x^2 + y^2 - r^2(t) = 0 \quad (4)$$

$$f_2(\dot{x},\dot{y},x,y,t) \equiv \dot{x}^2 + \dot{y}^2 - r^2(t) s^2(t) = 0$$

If in (4) both  $r(t)$  and  $s(t)$  vary independently, the system will represent a simultaneous AM and FM waveform with independent messages  $r(t)$  and  $s(t)$ .

## 2. ANALYSIS

### 2.1 Partitioning the Problem

Consider first the situation in which the analog computer is in the hold or reset mode. In this mode the correct stationary values of  $x$  and  $u$  must be generated. Since  $\phi$  is not a function of  $u$ , it is possible to generate  $x$  by programming the equations

$$\phi_1(x) = 0, \phi_2(x) = 0, \dots, \phi_m(x) = 0 \quad (5)$$

with  $n-m$  of the coordinates of  $x$  specified as initial conditions. This, of course, results in a system of  $m$  independent equations with  $m$  unknowns. Since these equations, in general, are nonlinear, the gradient method\* of programming is required to guarantee stability.

Turning now to the generation of  $u$ , the condition  $f = 0$  provides only  $n-m$  equations to be solved for  $n$  unknowns. To obtain the  $m$  additional equations an augmented velocity function vector ( $f'$ ) must be defined.

\* The gradient method is synonymous with least squares, steepest descent, or transpose matrix method.

In vector notation

$$f' = \frac{d\phi}{dt} + f \quad (6)$$

For the rigorous interpretation of (6), the original definitions of  $f$  and  $\phi$  must be expanded to  $n$ -dimensions, with the first  $m$  components of  $f$  being identically zero, and the last  $n-m$  components of  $\phi$  being identically zero. Written out:

$$f' \equiv \left[ \frac{d\phi_1}{dt}, \frac{d\phi_2}{dt}, \dots, \frac{d\phi_m}{dt}, f_{m+1}, f_{m+2}, \dots, f_n \right] \quad (7)$$

If in the first  $m$  coordinates of  $f'$  the substitution of  $u$  for  $\dot{x}$  is made, then

$$f' = 0 \quad (8)$$

represents a system of  $n$  equations in  $n$  unknowns  $u_1, u_2, \dots, u_n$ . The system (8) can be programmed by the gradient method and requires that  $x$  developed from (5) be inserted as a parameter. The conditions obtained in reset or hold then will be

$$\begin{aligned} \phi &= 0 \\ f' &= 0 \end{aligned} \quad (9)$$

(9) represents the partitioned system.

## 2.2 Closing the Switch

Refer now to figure 1, which is a simplified schematic illustrating the connections for the  $i$ -th component of  $x$  and  $u$ . In reset or hold, switch  $S$  is open and  $u$  and  $x$  assume their correct stationary values. It is reasonable to suppose that if  $S$  were closed, the resulting system would produce a very close approximation to the desired trajectory, provided the system maintains dynamic stability. The resulting computer differential equation will now be derived.

In accordance with the gradient method,

$$\text{let } v = -\text{grad}_x (\phi, \phi) = -\nabla_x (\phi, \phi) \quad (10)$$

$$a = -\text{grad}_u (f', f') = -\nabla_u (f', f') \quad (11)$$

From figure 1 we have

$$\dot{x} = Cu + Kv \quad (12)$$

$$\dot{u} = Ga \quad (13)$$



( $\alpha$  is the potentiometer setting, figure 1, and is the time scale factor) Differentiating (12) with respect to time and substituting (13)

$$\ddot{x} = \alpha G a + K \dot{v} \quad (14)$$

The system stability may be investigated by linearizing the system about some arbitrary operating point and examining the effect of small perturbations  $\delta x$  and  $\delta u$ . Operating on equations (12) and (14) with the variational operator  $\delta$

$$\frac{d(\delta x)}{dt} = \alpha \delta u + K \delta v \quad (15)$$

$$\frac{d^2(\delta x)}{dt^2} = \alpha G \delta a + K \frac{d(\delta v)}{dt} \quad (16)$$

It is now necessary to express  $\delta v$  and  $\delta a$  in terms of  $\delta x$ . Taking variations of both sides of (10)

$$\begin{aligned} \delta v_i = & \sum_{j=1}^n \left( -2 \sum_{k=1}^m \frac{\partial \phi_k}{\partial x_i} \frac{\partial \phi_k}{\partial x_j} \right) \delta x_j \\ & + \sum_{j=1}^n \left( -2 \sum_{k=1}^m \phi_k \frac{\partial^2 \phi_k}{\partial x_i \partial x_j} \right) \delta x_j \end{aligned} \quad (17)$$

Since variations are taken starting from an assumed equilibrium state  $\phi = 0$  the second term on the right in (17) vanishes. The matrix

$$A \equiv (a_{ij}) \equiv \left( 2 \sum_{k=1}^m \frac{\partial \phi_k}{\partial x_i} \frac{\partial \phi_k}{\partial x_j} \right)$$

is positive definite since it can be factored into the product of a matrix by its transpose; (17) then becomes

$$\delta v = -A \delta x \quad (18)$$

Note that since  $\phi$  is not a function of  $u$ , only the variation with respect to  $x$  had to be considered in (17). To find  $\delta a$ , take variations of both sides of (11), noting now that both  $\delta u$  and  $\delta x$  will contribute to  $\delta a$ .

$$\begin{aligned} \delta a_i = & \sum_{j=1}^n \left( -2 \sum_{k=1}^n \frac{\partial f'_k}{\partial u_i} \frac{\partial f'_k}{\partial u_j} \right) \delta u_j \\ & + \sum_{j=1}^n \left( -2 \sum_{k=1}^n \frac{\partial f'_k}{\partial u_i} \frac{\partial f'_k}{\partial x_j} \right) \delta x_j \end{aligned} \quad (19)$$

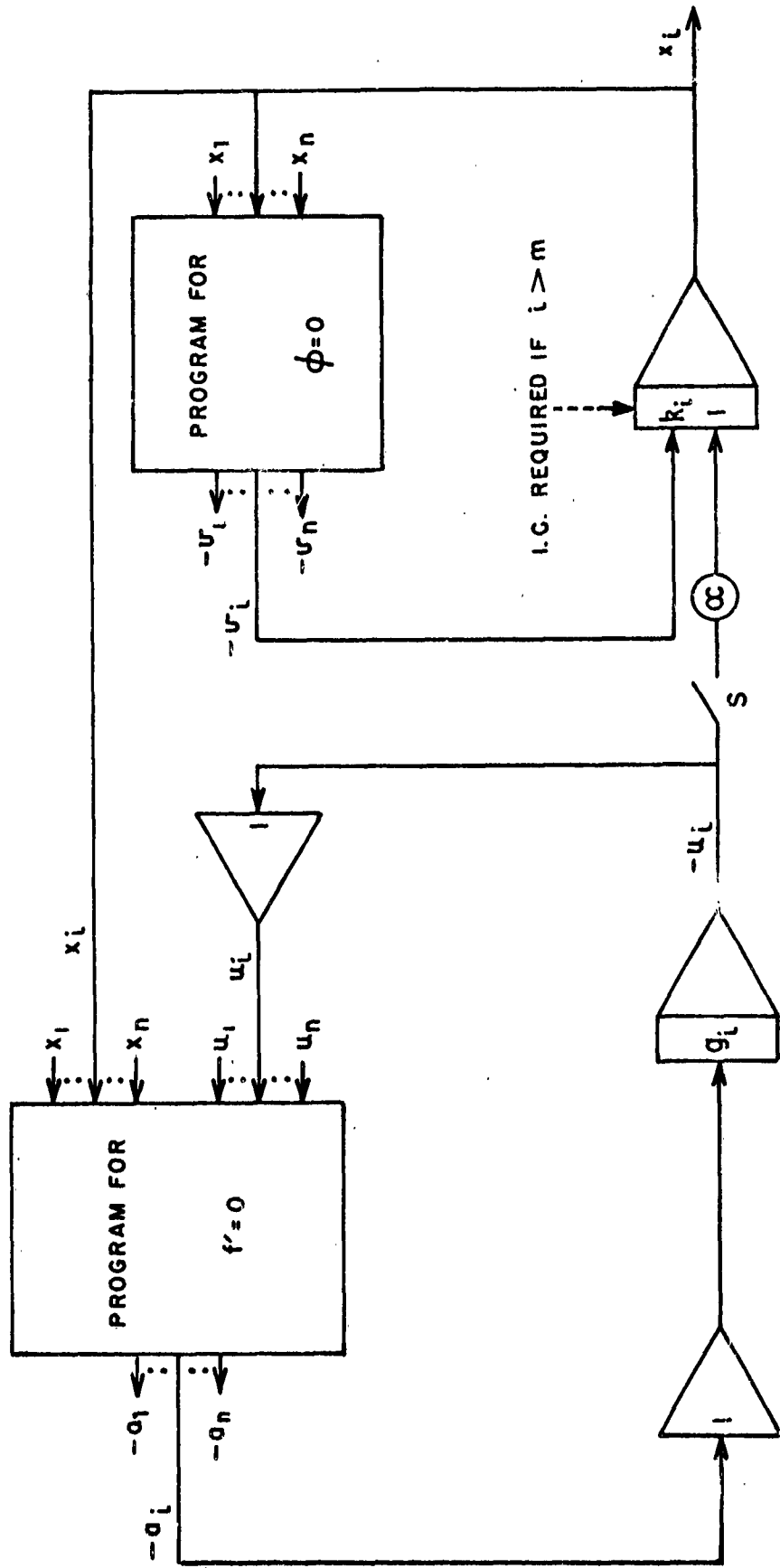


Figure 1. Simplified computer diagram showing connections for  $x_i, u_i$ .

In (19) terms involving second partial derivatives are absent because of the assumption that the initial conditions  $\phi = 0$  and  $f' = 0$  are satisfied.

Rewriting (19)

$$\delta a = -B \delta u - H \delta x \quad (20)$$

where

$$B \equiv (b_{ij}) \equiv \left( \sum_{k=1}^n 2 \frac{\partial f'_k}{\partial u_i} \frac{\partial f'_k}{\partial u_j} \right)$$

$$H \equiv (h_{ij}) \equiv \left( \sum_{k=1}^n 2 \frac{\partial f'_k}{\partial u_i} \frac{\partial f'_k}{\partial x_j} \right)$$

The matrix B is positive definite but the matrix H is not so restricted. Substituting (18) and (20) in (16)

$$\frac{d^2}{dt^2} (\delta x) + \alpha GB \delta u + \alpha GH \delta x + K \frac{d}{dt} (A \delta x) = 0 \quad (21)$$

$$\frac{d^2}{dt^2} (\delta x) + \alpha GB \delta u + \alpha GH \delta x + K \frac{dA}{dt} \delta x + KA \frac{d}{dt} (\delta x) = 0 \quad (22)$$

From (15) and (18)

$$\alpha \delta u = KA \delta x + \frac{d}{dt} (\delta x) \quad (23)$$

Substituting (23) in (22)

$$I \frac{d^2}{dt^2} (\delta x) + (KA + GB) \frac{d}{dt} (\delta x) + (GBKA + \alpha GH + K \frac{dA}{dt}) \delta x = 0 \quad (24)$$

(24) is the characteristic matrix differential equation of the system. Note that (I) the unity matrix is positive definite and (KA + GB) is positive definite, since it is the sum of two positive definite matrices KA and GB. Now it can be shown by an extension of the argument of Bellman (ref. 3) that (24) will be stable if the matrix sum

$$GBKA + \alpha GH + K \frac{dA}{dt} \quad (25)$$

can be expressed as the sum of a positive definite matrix and a skew symmetric matrix. The matrix product GBKA is positive definite and the matrix  $K \frac{dA}{dt}$  is symmetric. Let H be expressed as the sum of a symmetric part  $\tilde{H}$  and a skew symmetric part S

$$H = \tilde{H} + S \quad (26)$$

Let the notation  $A > B$  for two symmetric matrices denote the fact that  $A - B$  is positive definite. Then the condition for stability can be written

$$GBKA + \alpha G\tilde{H} + K \frac{dA}{dt} > 0 \quad (27)$$

or

$$GBKA > -\alpha G\tilde{H} - K \frac{dA}{dt} \quad (28)$$

It is now evident that if the term  $\alpha G\tilde{H}$  is causing instability, the scale factor  $\alpha$  must be decreased i.e., the trajectory is run slower than the real time case ( $\alpha = 1$ ). If the term  $K \frac{dA}{dt}$  is causing instability  $K$  can be decreased. If only the full set of velocity functions is given ( $f_1, f_2, \dots, f_n$ ), then  $A \equiv 0$  and the condition for stability becomes  $\tilde{H} > 0$ . This may be impossible to satisfy, in which case at least one function  $\phi_j$  must be found by integration from the system  $f = 0$ . Since at least  $\phi_j$  is available,  $A$  is reinstated and system stability is obtained as before.

### 3. CONCLUSION

It has been shown that the powerful negative gradient technique can be extended to successfully program trajectory problems as defined in the introduction. Computer stabilization may require time scaling (decreasing  $\alpha$ ) or reducing the gain  $K$ .

### ACKNOWLEDGMENT

The problem generalization and analysis in this report was inspired by the work of A. Hausner, who intuitively used this method to program a special problem proposed by the author.

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